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The Uniform Convergence of Special Standardized Distributions

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The Uniform Convergence of Special Standardized Distributions

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In this thesis, the uniform convergence of two special standardized distributions are examined. The Gamma distribution is standardized, and then the probability density function of the standardized Gamma distribution is shown to converge uniformly to the probability density function of the normal distribution. The F-distribution is standardized, and then the probability density function of the standardized F-distribution is shown to converge uniformly to the probability density function of the normal distribution. Along the way, some other interesting limits are observed.

Introduction

It was in the early nineteenth century that the idea of uniform convergence was first recognized by Augustin Cauchy in his book entitled *Analyse algébrique*, published in 1821. In this book, Cauchy proved that a convergent series of terms, each of which is continuous, converges to a continuous function [6]. By a careful examination of Cauchy's proof, Neils Abel discovered that the proof was invalid since a counterexample existed. As a result, the works of Philipp Seidel and George Stokes in the 1847 are usually linked to the development of the concept of uniform convergence. Neither of the mathematicians defined uniform convergence; instead they provided insight into types of convergence which are related to uniform convergence.

Cauchy corrected the error in his proof in a paper published in 1853 [6]. In the corrected proof, Cauchy did not use the term uniform convergence. However, he did impose on the convergent sequence of continuous functions the inequality conditions on remainders that characterize uniform convergence [6]. It was Karl Weierstrass who, in 1859, coined the term uniform convergence in the lectures he gave at the University of Berlin.

Due to the initial work of Cauchy, Seidel, Stokes, and Weierstrass, we now have an interesting analytic property. This property states that a sequence of functions $\{f_n(x)\}$ converges uniformly on a compact subset of the real numbers if there is a function $f(x)$ such that for each $\varepsilon > 0$ there is an integer N such that for $n \geq N$ $f_n(x)$ is within ε of $f(x)$ for all $x \in [-b, b]$ [1]. Our goal is to show that the probability density functions of the standardized gamma distribution and the standardized F-distribution have this analytic property. In particular, we shall demonstrate that the probability density functions of these standardized distributions converge uniformly on compact sets to the standard normal distribution.

Chapter One

Uniform Convergence of the Standardized Gamma Distribution

1.1 Introduction of the gamma distribution

The gamma distribution is a single-variable continuous distribution that is used to describe random variables bounded at one end. It is the appropriate model for the time required for a total of exactly α independent events to take place if events occur at a constant rate $1/\theta$; this suggests numerous applications. For example, if a part is ordered in lots of size α and the demand for the individual parts arises independently at a constant rate $1/\theta$ per week, then the time between lot depletions is a gamma variate. Similarly, system time to failure is gamma distributed if system failure occurs as soon as exactly α subfailures have taken place and if subfailures occur independently at a constant rate of $1/\theta$.

The probability density function (p.d.f.) of the gamma distribution with parameters α and θ is intimately connected with the Poisson process, which counts the number of occurrences per unit time. To illustrate this connection, let N_x be the number of occurrences in a unit of measure x which follows the Poisson process with mean $1/\theta$. Let X be the units of measurement needed until the α occurrences. Then

$$P[N_x < \alpha] = \sum_{n=0}^{\alpha-1} \frac{e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^n}{n!}$$

$$= P[X > x]$$

$$= 1 - F(x),$$

where $F(x)$ is the cumulative distribution function (c.d.f.) for the unit of measure until the α occurrences. This result implies that

$$F(x) = 1 - \sum_{n=0}^{\alpha-1} \frac{e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^n}{n!} .$$

Since the p.d.f. is the derivative of the c.d.f., we obtain the p.d.f. of the gamma distribution as follows:

$$f(x) = \frac{d}{dx}\left[1 - \sum_{n=0}^{\alpha-1} \frac{e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^n}{n!}\right]$$

$$= -\sum_{n=0}^{\alpha-1} \left[\frac{\frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1} e^{-\frac{x}{\theta}} - \frac{1}{\theta}\left(\frac{x}{\theta}\right)^n e^{-\frac{x}{\theta}}}{n!}\right]$$

$$= \frac{1}{\theta} e^{-\frac{x}{\theta}} - \frac{1}{\theta} e^{-\frac{x}{\theta}} \sum_{n=1}^{\alpha-1} \left[\frac{\left(\frac{x}{\theta}\right)^{n-1}}{(n-1)!} - \frac{\left(\frac{x}{\theta}\right)^n}{n!}\right]$$

$$= \frac{1}{\theta} e^{-\frac{x}{\theta}} - \frac{1}{\theta} e^{-\frac{x}{\theta}} \left[\sum_{n=1}^{\alpha-1} \frac{\left(\frac{x}{\theta}\right)^{n-1}}{(n-1)!} - \sum_{n=1}^{\alpha-1} \frac{\left(\frac{x}{\theta}\right)^n}{n!}\right]$$

$$= \frac{1}{\theta} e^{-\frac{x}{\theta}} - \frac{1}{\theta} e^{-\frac{x}{\theta}} \left(\left[1 + \sum_{n=2}^{\alpha-1} \frac{\left(\frac{x}{\theta}\right)^{n-1}}{(n-1)!}\right] - \left[\sum_{n=1}^{\alpha-2} \frac{\left(\frac{x}{\theta}\right)^n}{n!} + \frac{\left(\frac{x}{\theta}\right)^{\alpha-1}}{(\alpha-1)!}\right]\right)$$

$$= \frac{1}{\theta} e^{-\frac{x}{\theta}} - \frac{1}{\theta} e^{-\frac{x}{\theta}} \left(1 - \frac{\left(\frac{x}{\theta}\right)^{\alpha-1}}{(\alpha-1)!}\right)$$

$$= \frac{x^{\alpha-1}}{\theta^{\alpha}(\alpha-1)!} e^{-\frac{x}{\theta}},$$

for all $x \geq 0$. Since the random variable X is defined to be the units of measurement needed until the α occurrences with the average number of occurrences being $1/\theta$ per unit, X has a gamma distribution with parameters α and θ.

1.2 Standardizing the density of the gamma distribution

In general, the gamma distribution $X \sim \Gamma[\alpha, \theta]$ with shape parameter α and scale parameter θ is described by the probability density function

$$f_X(x) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\, x^{\alpha-1}\, e^{-\frac{x}{\theta}},$$

for $x \geq 0$, where the gamma function is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}\, e^{-t}\, dt.$$

A graph of the p.d.f. shows that there are three basic shapes for the p.d.f. of the gamma distribution. The three basic shapes are as follows: when $\theta = 1$ and $\alpha \in (0, 1)$, the graph is asymptotic to the y-axis (Figure 1); when $\theta = 1$ and $\alpha = 1$, the graph has a y-intercept and is strictly decreasing (Figure 2); and when $\theta = 1$ and $\alpha > 1$, the graph has a humpback shape (Figure 3).

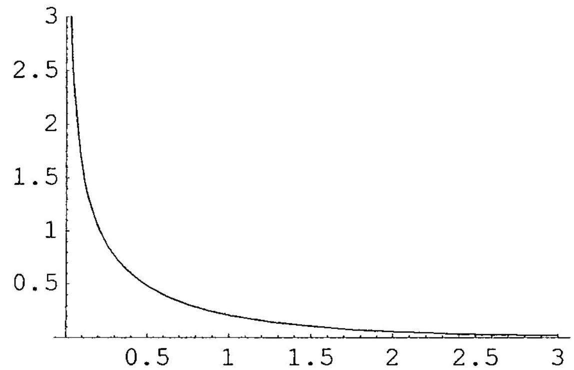

Figure 1: The gamma distribution when $\theta = 1$ and $\alpha = 0.5$.

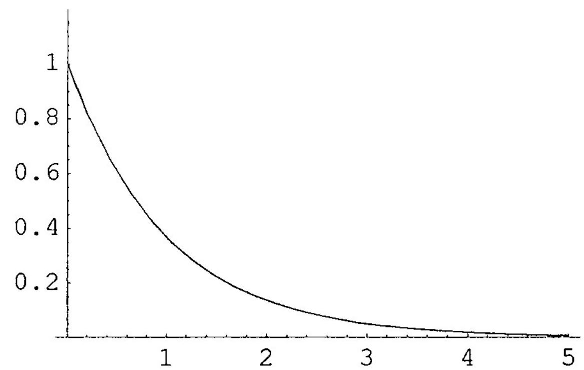

Figure 2: The gamma distribution when $\theta = 1$ and $\alpha = 1$.

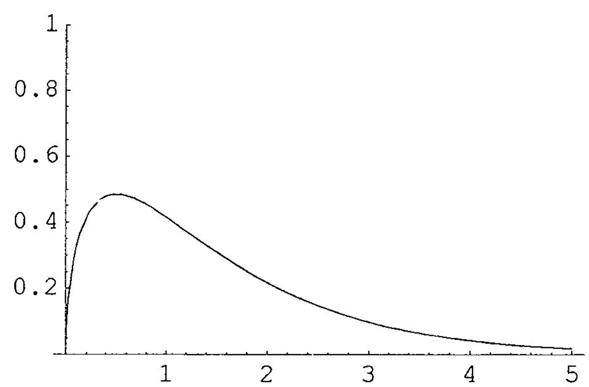

Figure 3: The gamma distribution when $\theta = 1$ and $\alpha = 1.5$.

The mean and standard deviation for the p.d.f. of the gamma distribution are obtained using the *rth* moment of X [9], which is:

$$E[X^r] = \int_0^\infty x^r f_X(x)\,dx$$

$$= \int_0^\infty \frac{x^{r+\alpha-1}}{\theta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\theta}}\,dx$$

$$= \frac{\theta^r \Gamma(\alpha+r)}{\Gamma(\alpha)}.$$

The mean is the first moment, and is given by

$$\mu = E[X]$$

$$= \frac{\theta\,\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$= \frac{\theta\,\alpha\,\Gamma(\alpha)}{\Gamma(\alpha)}$$

$$= \alpha\theta.$$

To determine the standard deviation, we first must compute the variance which is obtained by subtracting the square of the first moment from the second moment; thus the variance is given by

$$\sigma^2 = E[X^2] - (E[X])^2$$

$$= \frac{\theta^2 \Gamma(\alpha+2)}{\Gamma(\alpha)} - \left(\frac{\theta\,\Gamma(\alpha+1)}{\Gamma(\alpha)}\right)^2$$

$$= \frac{\theta^2 \alpha(\alpha+1)\,\Gamma(\alpha)}{\Gamma(\alpha)} - (\alpha\theta)^2$$

$$= \theta^2 \alpha(\alpha+1) - \theta^2 \alpha^2$$

$$= \alpha\theta^2.$$

Since the standard deviation σ is the square root of the variance, $\sigma = \sqrt{\alpha}\,\theta$.

The standardized gamma distribution $Y[\alpha, \theta]$ is obtained by subtracting the mean of $\Gamma[\alpha, \theta]$ and dividing by its standard deviation:

$$Y[\alpha, \theta] = \frac{\Gamma[\alpha,\theta]-\mu}{\sigma}$$

$$= \frac{\Gamma[\alpha,\theta]-\alpha\theta}{\sqrt{\alpha}\,\theta}.$$

In order to determine the p.d.f. of the standardized distribution, we shall apply the "c.d.f. technique"; that is, we write the cumulative distribution function G_Y of $Y[\alpha, \theta]$ is written in terms of the cumulative distribution function F_X of $\Gamma[\alpha, \theta]$ as follows:

$$G_Y(x) = P(Y[\alpha, \theta] \leq x)$$

$$= P\left(\frac{\Gamma[\alpha,\theta]-\mu}{\sigma} \leq x\right)$$

$$= P(\Gamma(\alpha, \theta) \leq \mu + \sigma\, x)$$

$$= F_X(\alpha\theta + \sqrt{\alpha}\,\theta\, x)$$

$$= F_X(\mu + \sigma\, x).$$

Next we take the first derivative to obtain the p.d.f. $g_Y(x)$. Thus the p.d.f. $g_Y(x)$ of $Y[\alpha, \theta]$ is given by:

$$g_Y(x) = \sigma\, f_X\,(\mu + \sigma\, x)$$

$$= \sqrt{\alpha}\;\theta\; f_X\left(\alpha\theta + \sqrt{\alpha}\;\theta\, x\right)$$

$$= \frac{\sqrt{\alpha}\;\theta}{\Gamma(\alpha)\,\theta^{\alpha}}\left(\alpha\theta + \sqrt{\alpha}\;\theta\, x\right)^{\alpha-1} e^{-\left(\frac{\alpha\,\theta+\sqrt{\alpha}\,\theta\, x}{\theta}\right)},$$

for $x \geq -\sqrt{\alpha}$. Since $\Gamma[\alpha, \theta]$ has range $[0, \infty)$, $Y[\alpha, \theta]$ has range $[-\sqrt{\alpha}, \infty)$ (Figure 4).

The p.d.f. of the standard normal distribution is defined for all x by $f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$. A graph of the standard normal distribution is the common "bell-shaped curve," which is symmetric about the origin (Figure 5).

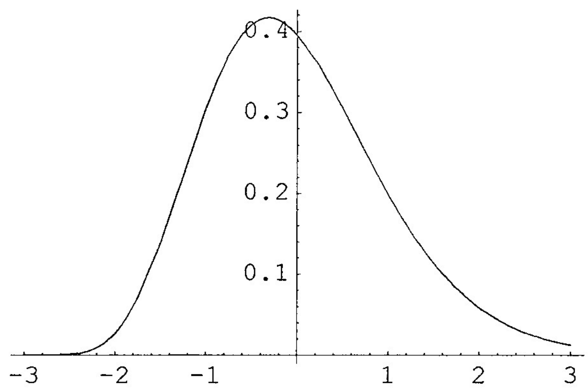

Figure 4: The Standardized $\Gamma[10, 10]$.

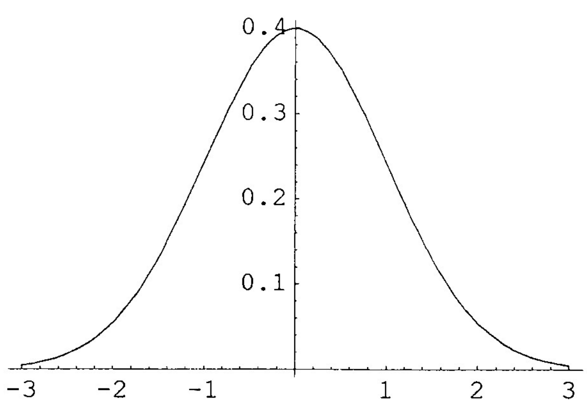

Figure 5: The standard normal distribution.

Henceforth, we shall hold the parameter θ constant and consider the class of functions $\{g_\alpha(x)\}$, which are the p.d.f.'s of $Y[\alpha, \theta]$. Our ultimate goal is to show that $\{g_\alpha(x)\}$ converges uniformly to $f(x)$ as α tends to infinity on any compact interval $[-b, b]$, where $b > 0$. We must note that in order for each function $g_\alpha(x)$ to be defined on all of $[-b, b]$, α may initially have to be large enough so that $-\sqrt{\alpha} < -b$.

1.3 Separating the functions $g_\alpha(x)$

In order to work with the functions $g_\alpha(x)$, we must divide them into three pieces. Thus we rewrite $g_\alpha(x)$ as follows

$$\begin{aligned} g_\alpha(x) &= \frac{\sqrt{\alpha}\,\theta}{\Gamma(\alpha)\,\theta^\alpha}\left(\alpha\theta + \sqrt{\alpha}\,\theta x\right)^{\alpha-1} e^{-\left(\frac{\alpha\,\theta + \sqrt{\alpha}\,\theta x}{\theta}\right)} \\ &= \frac{\sqrt{\alpha}\,\theta\,\alpha^{\alpha-1}\theta^{\alpha-1}}{\Gamma(\alpha)\,\theta^{\alpha}}\left(1 + \frac{x}{\sqrt{\alpha}}\right)^{\alpha-1} e^{-\alpha}\, e^{-\sqrt{\alpha}\,x} \\ &= \frac{\sqrt{\alpha}\,\alpha^{\alpha-1}e^{-\alpha}}{\Gamma(\alpha)}\left(1 + \frac{x}{\sqrt{\alpha}}\right)^{-1}\left(1 + \frac{x}{\sqrt{\alpha}}\right)^{\alpha} e^{-\sqrt{\alpha}\,x}. \end{aligned}$$

Now we collect the constant terms and let

$$C_\alpha = \frac{\sqrt{\alpha}\,\alpha^{\alpha-1}e^{-\alpha}}{\Gamma(\alpha)}. \tag{1.1}$$

Next we define $h_\alpha(x)$ and $k_\alpha(x)$ by

$$h_\alpha(x) = \frac{1}{1+\frac{x}{\sqrt{\alpha}}},$$

and

$$k_\alpha(x) = \frac{\left(1+\frac{x}{\sqrt{\alpha}}\right)^\alpha}{e^{\sqrt{\alpha}\,x}}.$$

Then $g_\alpha(x) = C_\alpha\ h_\alpha(x)\ k_\alpha(x)$. Lastly, we separate $f(x)$ by letting $C = \frac{1}{\sqrt{2\pi}}$, and $g(x) = e^{-x^2/2}$.

We will first show that $\{g_\alpha(x)\}$ converges pointwise to $f(x)$ by showing the convergence of the separate pieces.

1.4 The limit of the constants C_α

Since each term in C_α is continuous and monotone, it suffices to allow α to vary over the positive integers. Then to evaluate the limit of the constants C_α, we shall express $\Gamma(\alpha)$ in terms of factorials. Since $\Gamma(\alpha) = (\alpha-1)!$ for positive integers α, Equation (1.1) can now be expressed as

$$C_\alpha = \frac{\sqrt{\alpha}\ \alpha^{\alpha-1}\ e^{-\alpha}}{(\alpha-1)!}. \tag{1.2}$$

To evaluate the limit of $\{C_\alpha\}$, we shall apply Stirling's Formula [7] which states that for integers α,

$$\lim_{\alpha\to\infty} \frac{\alpha!}{\sqrt{2\pi\alpha}\left(\frac{\alpha}{e}\right)^\alpha} = 1.$$

We can therefore replace the expression $\alpha!$ with $\sqrt{2\pi\alpha}\left(\frac{\alpha}{e}\right)^\alpha$ when taking the limit. Thus, we replace $(\alpha-1)!$ in Equation (1.2) with $\sqrt{2\pi(\alpha-1)}\left(\frac{\alpha-1}{e}\right)^{\alpha-1}$. We are now ready to take the limit of C_α in the following theorem.

Theorem 1.1. $\lim_{\alpha\to\infty} C_\alpha = \frac{1}{\sqrt{2\pi}}$.

Proof.

$$\lim_{\alpha\to\infty} C_\alpha = \lim_{\alpha\to\infty} \frac{\sqrt{\alpha}\,\alpha^{\alpha-1}e^{-\alpha}}{\sqrt{2\pi(\alpha-1)}\left(\frac{\alpha-1}{e}\right)^{\alpha-1}}$$

$$= \lim_{\alpha\to\infty} \frac{1}{e\sqrt{2\pi}}\sqrt{\frac{\alpha}{(\alpha-1)}}\left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1}$$

$$= \frac{1}{e\sqrt{2\pi}}\lim_{\alpha\to\infty}\sqrt{\frac{\alpha}{(\alpha-1)}}\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{\alpha}{\alpha-1}\right)^{\alpha}$$

$$= \frac{1}{e\sqrt{2\pi}}\lim_{\alpha\to\infty}\left(\frac{\alpha}{\alpha-1}\right)^{\alpha}$$

$$= \frac{1}{e\sqrt{2\pi}}\lim_{\alpha\to\infty}\frac{1}{\left(1-\frac{1}{\alpha}\right)^{\alpha}}$$

$$= \frac{1}{e\sqrt{2\pi}}\,\frac{1}{e^{-1}}$$

$$= \frac{1}{\sqrt{2\pi}}.$$

Since the limit of C_α as α tends to infinity is $\frac{1}{\sqrt{2\pi}}$, the sequence $\{C_\alpha\}$ converges to $C = \frac{1}{\sqrt{2\pi}}$. ■

1.5 Pointwise convergence of $h_\alpha(x)k_\alpha(x)$

We shall now show that $h_\alpha(x)\,k_\alpha(x)$ converges pointwise to $g(x)$ by first showing the convergence of $h_\alpha(x)$ and $k_\alpha(x)$. For the sequence $\{h_\alpha(x)\}$, it is clear that for $x > -\sqrt{\alpha}$,

$$\lim_{\alpha\to\infty}\frac{1}{1+\frac{x}{\sqrt{\alpha}}} = 1.$$

There are singularities at $x = -\sqrt{\alpha}$; however, the singularities are avoided on the interval $[-b, \infty)$ by initially choosing α large enough. Thus, $\{h_\alpha(x)\}$ converges pointwise to 1 on $[-b, \infty)$. We next show that $\lim_{\alpha\to\infty} k_\alpha(x) = g(x)$.

Theorem 1.2. For any interval $[-b, \infty)$, $\lim_{\alpha\to\infty} \dfrac{\left(1+\frac{x}{\sqrt{\alpha}}\right)^\alpha}{e^{\sqrt{\alpha}\,x}} = e^{-x^2/2}$.

Proof. The theorem is true for $x = 0$, since $\lim_{\alpha\to\infty} 1^{-\alpha} = e^0$. In order for the functions to be defined for other x, we must first let α be large enough so that $-\sqrt{\alpha} < -b$. To evaluate the limit of k_α as α tends to infinity, it will suffice to evaluate the limit of the natural logarithm of k_α, which exists for all x since we have avoided the values of x at $-\sqrt{\alpha}$. Simplifying the natural logarithm of $k_\alpha(x)$ before hand, we have:

$$\ln \frac{\left(1+\frac{x}{\sqrt{\alpha}}\right)^\alpha}{e^{\sqrt{\alpha}\,x}} = \alpha \ln\left(1 + \frac{x}{\sqrt{\alpha}}\right) - \ln\left(e^{\sqrt{\alpha}\,x}\right)$$

$$= \frac{\ln\left(1+\frac{x}{\sqrt{\alpha}}\right) - \frac{x}{\sqrt{\alpha}}}{\frac{1}{\alpha}}.$$

Now taking the limit as α tends to infinity, we have the indeterminate expression 0/0; thus we apply L'Hopital's Rule. Taking the derivatives of both the numerator and the denominator with respect to α, we obtain:

$$\frac{\frac{d}{d\alpha}\left[\ln\left(1+\frac{x}{\sqrt{\alpha}}\right)-\frac{x}{\sqrt{\alpha}}\right]}{\frac{d}{d\alpha}\left[\frac{1}{\alpha}\right]} = \frac{\frac{1}{1+\frac{x}{\sqrt{\alpha}}}\left(\frac{-x}{2\,\alpha^{3/2}}\right)+\left(\frac{x}{2\,\alpha^{3/2}}\right)}{\frac{-1}{\alpha^2}}$$

$$= \frac{\frac{x}{2\,\alpha^{3/2}}\left(\frac{x}{x+\sqrt{\alpha}}\right)}{\frac{-1}{\alpha^2}}$$

$$= \frac{\sqrt{\alpha}\;x^2}{-2\left(x+\sqrt{\alpha}\right)}.$$

Now taking the limit as α tends to infinity we have:

$$\lim_{\alpha\to\infty}\frac{\sqrt{\alpha}\;x^2}{-2\left(x+\sqrt{\alpha}\right)} = \frac{-x^2}{2}\;\frac{1}{\frac{x}{\sqrt{\alpha}}+1}$$

$$= \frac{-x^2}{2}.$$

Since we began by taking the natural logarithm of k_α, we find that

$$\lim_{\alpha\to\infty}\frac{\left(1+\frac{x}{\sqrt{\alpha}}\right)^{\alpha}}{e^{\sqrt{\alpha}\;x}} = e^{-x^2/2}.$$

Therefore, $\{k_\alpha(x)\}$ converges pointwise to $e^{-x^2/2}$ on $[-b, \infty)$ [See 8 for a similar argument]. ■

We have shown that the separate pieces of $\{g_\alpha(x)\}$ converge pointwise to their respective limits on $[-b, \infty)$. Thus the product still converges pointwise, which allows us to state the following result.

Corollary 1.1. For any x in the interval $[-b, \infty)$,

$$\lim_{\alpha\to\infty} \frac{\sqrt{\alpha}\,\alpha^{\alpha-1}e^{-\alpha}}{\Gamma(\alpha)}\left(1+\frac{x}{\sqrt{\alpha}}\right)^{-1}\left(1+\frac{x}{\sqrt{\alpha}}\right)^{\alpha} e^{-\sqrt{\alpha}\,x} = \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}.$$

1.6 Uniform convergence of the standardized gamma distribution

Since pointwise convergence does not imply uniform convergence, we will show that on $[-b, b]$ the sequence of functions $\{h_\alpha(x)\}$ converges uniformly to 1, the sequence of functions $\{k_\alpha(x)\}$ converges uniformly to $e^{-x^2/2}$, and the product $\{h_\alpha(x)\,k_\alpha(x)\}$ converges uniformly to $e^{-x^2/2}$. We will then conclude this chapter by proving that the sequence of functions $\{g_\alpha(x)\} = \{C_\alpha\, h_\alpha(x)\, k_\alpha(x)\}$ converges uniformly to $\frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}$ on $[-b, b]$.

We have seen that the separate pieces of $\{g_\alpha(x)\}$ converge pointwise to their respective limits. However, to show that a sequence of functions converges uniformly to $f(x)$, we must show that for each $\varepsilon > 0$ there exists an integer N such that for $\alpha \geq N$ $f_\alpha(x)$ is within ε of $f(x)$ for all $x \in [-b, b]$ [1]. For the sequence of functions $\{h_\alpha(x)\}$, this result is easily shown in the following lemma.

Lemma 1.1. On any interval $[-b, b]$, the sequence of functions $\left\{\frac{1}{1+\frac{x}{\sqrt{\alpha}}}\right\}$ converges uniformly to 1.

Proof. Let $\varepsilon > 0$ be given. Since $\lim\limits_{\alpha\to\infty} \dfrac{1}{1+\frac{b}{\sqrt{\alpha}}} = 1$ and $\lim\limits_{\alpha\to\infty} \dfrac{1}{1-\frac{b}{\sqrt{\alpha}}} = 1$, there exists an integer N such that if $\alpha \geq N$, then $-\sqrt{\alpha} < -b$, $\left| \dfrac{1}{1+\frac{b}{\sqrt{\alpha}}} - 1 \right| < \varepsilon$, and $\left| \dfrac{1}{1-\frac{b}{\sqrt{\alpha}}} - 1 \right| < \varepsilon$. Then for all $\alpha \geq N$,

$$1 - \varepsilon < \frac{1}{1+\frac{b}{\sqrt{\alpha}}} \leq \frac{1}{1+\frac{x}{\sqrt{\alpha}}} \leq 1,$$

for $0 \leq x \leq b$; and

$$1 \leq \frac{1}{1+\frac{x}{\sqrt{\alpha}}} \leq \frac{1}{1-\frac{b}{\sqrt{\alpha}}} < 1 + \varepsilon,$$

for $-b \leq x \leq 0$. Thus for all $x \in [-b, b]$, $\dfrac{1}{1+\frac{x}{\sqrt{\alpha}}}$ is within ε of 1 if $\alpha \geq N$ (see 8 for a similar argument). Therefore the sequence of functions $\{h_\alpha(x)\}$ converges uniformly to 1. ■

Now we consider the sequence of functions $\{k_\alpha(x)\}$.

Lemma 1.2. On any interval $[-b, b]$, the sequence of functions $\left\{ \dfrac{\left(1+\frac{x}{\sqrt{\alpha}}\right)^{\alpha}}{e^{\sqrt{\alpha}\,x}} \right\}$ converges uniformly to $e^{-x^2/2}$.

Proof. At the end of Section 1.2, we stated that we are initially starting our sequences with α large enough so that $-\sqrt{\alpha} < -b$. Thus we have $-1 < \dfrac{x}{\sqrt{\alpha}} < 1$ for all $x \in [-b, b]$.

We claim that the sequence of functions $\{k_\alpha(x)\}$ is decreasing for $x > 0$ and increasing for $x < 0$. To demonstrate this result it will suffice to show that the sequence $\{b_\alpha(x)\} = \{\ln(k_\alpha(x))\}$ is decreasing for $x > 0$ and increasing for $x < 0$.

The Maclaurin series expansion of $\ln(1 + x)$, for $-1 < x < 1$ is given by:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, x^k}{k}.$$

Thus,

$$b_\alpha(x) = \alpha \ln\left(1 + \frac{x}{\sqrt{\alpha}}\right) - x\sqrt{\alpha}$$

$$= \alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, x^k}{k\, \alpha^{k/2}} - x\sqrt{\alpha}$$

$$= \alpha\left(\frac{x}{\alpha^{1/2}} - \frac{x^2}{2\alpha} + \frac{x^3}{3\alpha^{3/2}} - \frac{x^4}{4\alpha^2} + \frac{x^5}{5\alpha^{5/2}} - \ldots\right) - x\sqrt{\alpha}$$

$$= -\left(\frac{x^2}{2} - \frac{x^3}{3\alpha^{1/2}} + \frac{x^4}{4\alpha} - \frac{x^5}{5\alpha^{3/2}} + \ldots\right).$$

Now for $x < 0$, each term in the expression $\left(\frac{x^2}{2} - \frac{x^3}{3\alpha^{1/2}} + \frac{x^4}{4\alpha} - \frac{x^5}{5\alpha^{3/2}} + \ldots\right)$ is positive since the odd powers of x have negative coefficients. Thus as α increases, the terms become smaller, so the terms in the expression decrease as a sequence in α. Thus the sequence of functions $\{b_\alpha(x)\}$ is an increasing sequence for $x < 0$.

In order to show that $\{b_\alpha(x)\}$ decreases for $x > 0$, we will show that its derivative with respect to α is negative. When we take the derivative of $b_\alpha(x)$ and use the Maclaurin series, we obtain:

$$\frac{d(b_\alpha(x))}{d\alpha} = \ln\left(1 + \frac{x}{\sqrt{\alpha}}\right) + \alpha\left(\frac{1}{1+\frac{x}{\sqrt{\alpha}}}\right)\left(\frac{-x}{2\,\alpha^{3/2}}\right) - \left(\frac{x}{2\sqrt{\alpha}}\right)$$

$$= \ln\left(1 + \frac{x}{\sqrt{\alpha}}\right) - \frac{x}{2\left(x+\sqrt{\alpha}\right)} - \left(\frac{x}{2\sqrt{\alpha}}\right)$$

$$= \left(\frac{x}{\alpha^{1/2}} - \frac{x^2}{2\,\alpha} + \frac{x^3}{3\,\alpha^{3/2}} - \frac{x^4}{4\,\alpha^2} + \ldots\right) - \frac{x}{2\left(x+\sqrt{\alpha}\right)} - \frac{x}{2\sqrt{\alpha}}.$$

The terms $\frac{x^k}{k\,\alpha^{k/2}}$ in the alternating series are decreasing with k, since $0 < \frac{x}{\sqrt{\alpha}} < 1$. Thus the series converges; but more importantly, the entire sum is less than any partial sum that is stopped after a negative term. Therefore for $x > 0$,

$$\frac{d(b_\alpha(x))}{d\alpha} < \left(\frac{x}{\alpha^{1/2}} - \frac{x^2}{2\,\alpha}\right) - \frac{x}{2\left(x+\sqrt{\alpha}\right)} - \frac{x}{2\sqrt{\alpha}}$$

$$= \frac{-x^3}{2\,\alpha\left(x+\sqrt{\alpha}\right)}$$

$$< 0.$$

Therefore as sequences in α, $\{b_\alpha(x)\}$ is decreasing for $x > 0$, constant for $x = 0$, and increasing for $x < 0$. Thus, $\{k_\alpha(x)\}$ is decreasing for $x > 0$, constant for $x = 0$, and increasing for $x < 0$.

If a monotone sequence of continuous functions converges pointwise to a continuous function on a compact set, then by Dini's theorem [4] the sequence converges uniformly on any compact interval. Hence, the sequence $\{k_\alpha(x)\}$ converges uniformly to $e^{-x^2/2}$ on $[-b, 0]$ and it converges uniformly to $e^{-x^2/2}$ on $[0, b]$. Now for any $\varepsilon > 0$, we choose α large enough so that $\{k_\alpha(x)\}$ is within ε of $e^{-x^2/2}$ for all $x \in [-b, b]$. Therefore the sequence $\{k_\alpha(x)\}$ converges uniformly to $e^{-x^2/2}$ on $[-b, b]$ [see 8 for a similar argument]. ■

It is important to note that in general, the product of two uniformly convergent sequences does not necessarily converge uniformly. However if the two uniformly convergent sequences are bounded, we can prove the uniform convergence of the product [1]. Thus we have the following lemma.

Lemma 1.3. On any interval $[-b, b]$, the sequence of functions $\left\{\frac{1}{1+\frac{x}{\sqrt{\alpha}}} \frac{\left(1+\frac{x}{\sqrt{\alpha}}\right)^\alpha}{e^{\sqrt{\alpha}\, x}}\right\}$ converges uniformly to $e^{-x^2/2}$.

Proof. Let $h_\alpha(x) = \dfrac{1}{1+\frac{x}{\sqrt{\alpha}}}$, $k_\alpha(x) = \dfrac{\left(1+\frac{x}{\sqrt{\alpha}}\right)^\alpha}{e^{\sqrt{\alpha}\, x}}$, and $g(x) = e^{-x^2/2}$.

Then $|g(x)| \leq 1$ for all x. As in Lemma 1.1, there exists an N such that if $\alpha \geq N$, then $-\sqrt{\alpha} < -b$. Thus for $\alpha \geq N$, $h_\alpha(x)$ is continuous, pointwise convergent, and hence bounded on the compact interval $[-b, b]$. Now we have a uniformly convergent sequence of bounded functions. Thus $\{h_\alpha(x)\}$ is uniformly bounded. Hence, there exists an M such that $|h_\alpha(x)| \leq M$ for all $x \in [-b, b]$ and all $\alpha \geq N$ [1].

Now let $\varepsilon > 0$ be given. By Lemmas 1.1 and 1.2 there exists an $N_1 \geq N$ such that if $\alpha \geq N$, then $\left|k_\alpha(x) - g(x)\right| < \frac{\varepsilon}{2M}$ and $\left|h_\alpha(x) - 1\right| < \frac{\varepsilon}{2}$ for all $x \in [-b, b]$. Thus for $\alpha \geq N$, and all $x \in [-b, b]$, we have the following:

$$
\begin{aligned}
|\, h_\alpha(x)\, k_\alpha(x) - g(x) \,| &= |\, h_\alpha(x)\, k_\alpha(x) - h_\alpha(x)\, g(x) + h_\alpha(x)\, g(x) - g(x) \,| \\
&\leq |\, h_\alpha(x)\, k_\alpha(x) - h_\alpha(x)\, g(x) \,| + |\, h_\alpha(x)\, g(x) - g(x) \,| \\
&= |\, h_\alpha(x) \,| \;\; |\, k_\alpha(x) - g(x) \,| + \; |\, h_\alpha(x) - 1 \,| \;\; |\, g(x) \,| \\
&\leq (M)\, |\, k_\alpha(x) - g(x) \,| + |\, h_\alpha(x) - 1 \,| \; (1) \\
&< (M)\left(\tfrac{\varepsilon}{2M}\right) + \left(\tfrac{\varepsilon}{2}\right)(1) \\
&= \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Therfore the sequence of functions $\{h_\alpha(x)\, k_\alpha(x)\}$ converges uniformly to $e^{-x^2/2}$ on $[-b, b]$. ■

We are now ready to prove our main result.

Theorem 1.4. On any interval $[-b, b]$, the probability density functions $\{g_\alpha(x)\}$ of the standardized gamma distributions converge uniformly to the probability density function $f(x)$ of the standard normal distribution.

Proof. We write $g_\alpha(x) = C_\alpha\; h_\alpha(x)\; k_\alpha(x)$ and $f(x) = C\, g(x)$, where the product of the sequence of functions $\{h_\alpha(x)\, k_\alpha(x)\}$ converges uniformly to $g(x)$ and the sequence of constants $\{C_\alpha\}$ converges to C. The sequence $\{C_\alpha\}$ is bounded by some constant L, since $\{C_\alpha\}$ converges to C [4]. As noted in Lemma 1.3, $g(x)$ is bounded by 1. Now given $\varepsilon > 0$, there exists an N such that if $\alpha \geq N$, then $\left| C_\alpha - C \right| < \frac{\varepsilon}{2}$ and $\left| h_\alpha(x)\, k_\alpha(x) - g(x) \right| < \frac{\varepsilon}{2L}$ for all $x \in [-b, b]$. Then for $\alpha \geq N$, and all $x \in [-b, b]$, we have the following:

$$
\begin{aligned}
|\, g_\alpha(x) - f(x) \,| &= |\, C_\alpha\, h_\alpha(x)\, k_\alpha(x) - C\, g(x) \,| \\
&= |\, C_\alpha\, h_\alpha(x)\, k_\alpha(x) - C_\alpha\, g(x) + C_\alpha\, g(x) - C\, g(x) \,| \\
&\leq |\, C_\alpha\, h_\alpha(x)\, k_\alpha(x) - C_\alpha\, g(x) \,| + |\, C_\alpha\, g(x) - C\, g(x) \,| \\
&= |\, C_\alpha \,|\ |\, h_\alpha(x)\, k_\alpha(x) - g(x) \,| + |\, C_\alpha - C \,|\ |\, g(x) \,| \\
&< (L)\left(\tfrac{\varepsilon}{2L}\right) + \left(\tfrac{\varepsilon}{2}\right)(1) \\
&= \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Therefore the p.d.f.'s $\{g_\alpha(x)\}$ of the standardized gamma distributions converge uniformly on $[-b, b]$ to the p.d.f $f(x)$ of the standard normal distribution. ∎

Chapter Two

Uniform Convergence of the Standardized F-Distribution

2.1 Introduction of the F-Distribution

The F-distribution is a continuous statistical distribution that is used in making tests of hypotheses to determine the validity of the assumption of identical standard deviations of two normally distributed populations. It is also the distribution on which the whole analysis of variance procedure is based.

The F-distribution has a natural relationship with the chi-square distribution. If U and V are both chi-square with m and n degrees of freedom respectively, then the statistic

$$\mathrm{X} = \frac{U/m}{V/n}$$

is F-distributed. The two parameters m and n are the numerator and denominator degrees of freedom; that is, m and n are the numbers of independent pieces of information used to calculate U and V, respectively.

Since U and V are independent, their joint density is given by

$$f(u, v) = \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)2^{(m+n)/2}} u^{\frac{m-2}{2}} v^{\frac{n-2}{2}} e^{-\frac{u+v}{2}},$$

for u, $v > 0$. To find the distribution function of X, we must employ a change of variables; that is, we let u and v be transformed to new variables x and w by the transformations $u = \frac{m w x}{n}$ and $v = w$.

Then

$$\frac{\partial u}{\partial x} = \frac{m w}{n}, \qquad \frac{\partial u}{\partial w} = \frac{m x}{n},$$

$$\frac{\partial v}{\partial x} = 0, \text{ and} \qquad \frac{\partial v}{\partial w} = 1,$$

so that the Jacobian of the inverse transformation from x and w to u and v is $\frac{m w}{n}$.

Hence, the joint density function of X and W is

$$g(x, w) = \frac{1}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})2^{(m+n)/2}} (\frac{m w x}{n})^{\frac{m-2}{2}} w^{\frac{n-2}{2}} e^{-\frac{(m w x)/n+w}{2}} (\frac{m}{n}) w$$

$$= \frac{1}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} (\frac{m}{n})^{\frac{m}{2}} x^{\frac{m-2}{2}} \left(\frac{1}{2}\right)\left(\frac{w}{2}\right)^{\frac{m+n-2}{2}} e^{\frac{-(wn+w m x)/n}{2}}.$$

We now integrate out w to obtain

$$h(x) = \frac{1}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} (\frac{m}{n})^{\frac{m}{2}} x^{\frac{m-2}{2}} \left(\frac{1}{2}\right) \int_0^{\infty} \left(\frac{w}{2}\right)^{\frac{m+n-2}{2}} e^{\frac{-w(n+ m x)/n}{2}} dw. \tag{2.1}$$

Let

$$A = \int_0^{\infty} \left(\frac{w}{2}\right)^{\frac{m+n-2}{2}} e^{\frac{-w(n+ m x)/n}{2}} dw. \tag{2.2}$$

Next, let

$$z = \frac{w}{2} (1 + \frac{m}{n} x),$$

and

$$\alpha = \frac{m+n}{2} - 1.$$

Then

$$\frac{w}{2} = (1 + \frac{m}{n}x)^{-1} z,$$

and

$$\left(\frac{w}{2}\right)^{\alpha} = (1 + \frac{m}{n}x)^{\frac{-(m+n-2)}{2}} z^{\alpha}.$$

Also,

$$d\left(\frac{w}{2}\right) = (1 + \frac{m}{n}x)^{-1} dz.$$

With these substitutions, Equation (2.2) becomes:

$$A = (1 + \frac{m}{n}x)^{\frac{-(m+n)}{2}} \int_0^{\infty} z^{\alpha} e^{-z} dz$$

$$= (1 + \frac{m}{n}x)^{\frac{-(m+n)}{2}} \Gamma\left(\frac{m+n}{2}\right).$$

Consequently, returning to Equation (2.1), we finally obtain the density function for X:

$$h(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{\frac{m}{2}} (1 + \frac{m}{n}x)^{-\frac{m+n}{2}} x^{\frac{m-2}{2}},$$

for $x > 0$. A random variable having the function $h(x)$ as its density function is said to follow the F-distribution with m and n degrees of freedom.

2.2 Standardizing the density of the F-distribution

In general, the F-distribution with m and n degrees of freedom $X \sim F[m, n]$ is described by the probability density function

$$f_X(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)\left(\frac{m}{n}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \; x^{\frac{m}{2}-1}\left(1+\frac{m}{n}\,x\right)^{-\frac{m+n}{2}}$$

for $x \geq 0$, $m > 0$, and $n > 0$, where the gamma function is defined for $a > 0$ by

$$\Gamma(a) = \int_0^{\infty} t^{a-1}\, e^{-t}\, dt.$$

A graph of the p.d.f. shows that there are two basic shapes for the p.d.f. of the F-distribution. The two basic shapes are as follows: when $n = 1$ and $m \leq 2$, the graph is asymptotic to the y-axis (Figure 6); when $n = 1$ and $m > 2$, the graph has a humpback shape (Figure 7).

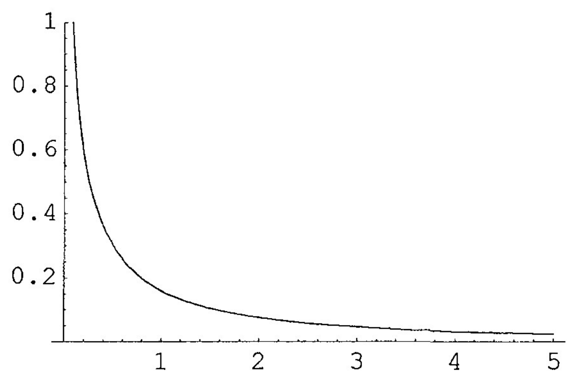

Figure 6: The F-distribution when $n = 1$ and $m = 1$.

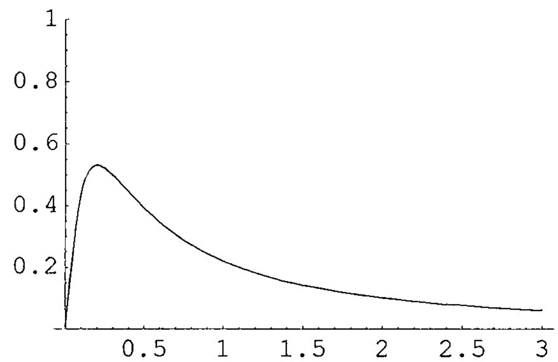

Figure 7: The F-distribution when $n = 1$ and $m = 5$.

The mean and standard deviation of the F-distribution are obtained using the *rth* moment of X [9] which is

$$E[X^r] = (\tfrac{n}{m})^r \frac{\Gamma(r+\frac{m}{2})\Gamma(\frac{n}{2}-r)}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})},$$

for $n > 2r$. The mean is the first moment, and is given by

$$\begin{aligned}
\mu &= E[X] \\
&= (\tfrac{n}{m}) \frac{\Gamma(1+\frac{m}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \\
&= (\tfrac{n}{m}) \frac{(\frac{m}{2})\Gamma(\frac{m}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{m}{2})(\frac{n}{2}-1)\Gamma(\frac{n}{2}-1)} \\
&= (\tfrac{n}{m}) \frac{m/2}{(n-2)/2} \\
&= \frac{n}{n-2},
\end{aligned}$$

for $n > 2$. Since the standard deviation is the square root of the variance, we must first find the variance. The variance is given by

$$\sigma^2 = E[X^2] - (E[X])^2$$

$$= \left(\frac{n}{m}\right)^2 \frac{\Gamma\left(2+\frac{m}{2}\right)\Gamma\left(\frac{n}{2}-2\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} - \left(\left(\frac{n}{m}\right) \frac{\Gamma\left(1+\frac{m}{2}\right)\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\right)^2$$

$$= \left(\frac{n}{m}\right)^2 \frac{(m/2)\,[(m/2)+1]}{[(n/2)-1]\,[(n/2)-2]} - \left(\left(\frac{n}{m}\right) \frac{m/2}{(n-2)/2}\right)^2$$

$$= \left(\frac{n}{m}\right)^2 \frac{m\,(m+2)}{(n-2)\,(n-4)} - \left(\frac{n}{n-2}\right)^2$$

$$= \frac{2\,n^2(m+n-2)}{m(n-2)^2\,(n-4)},$$

for $n > 4$ [see 9 for a similar argument]. Thus the standard deviation of the F-distribution is given by

$$\sigma = \sqrt{\frac{2\,n^2(m+n-2)}{m(n-2)^2\,(n-4)}}$$

$$= \frac{n}{n-2}\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\,.$$

The standardized F-distribution $Y[m, n]$ is obtained by

$$Y[m, n] = \frac{F[m,n]-\frac{n}{n-2}}{\frac{n}{n-2}\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}}\,.$$

In order to determine the p.d.f. of the standardized distribution, the "c.d.f. technique" is applied; that is, the cumulative distribution function G_Y of $Y[m, n]$ is written in terms of the cumulative distribution function F_X of $X \sim F[m, n]$. Then the first derivative of G_Y is taken to obtain the p.d.f. $g_Y(x)$ [see Section 1.2]. Since $F[m, n]$ has range $[0, \infty)$, $Y[m, n]$ has range $[-\sqrt{\frac{m(n-4)}{2\,(m+n-2)}}, \infty)$ (Figure 8). The p.d.f. of $Y[m, n]$ is then given by

$$g_Y(x) = \sigma\, f_X(\mu + \sigma x)$$

$$= \sqrt{\frac{2\,n^2(m+n-2)}{m(n-2)^2\,(n-4)}}\ \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\left(\frac{n}{n-2}+\sqrt{\frac{2\,n^2(m+n-2)}{m(n-2)^2\,(n-4)}}\ x\right)^{\frac{m}{2}-1} *$$

$$(\tfrac{m}{n})^{\frac{m}{2}}\left(1+\frac{m}{n}\left(\frac{n}{n-2}+\sqrt{\frac{2\,n^2(m+n-2)}{m(n-2)^2\,(n-4)}}\ x\right)\right)^{-\frac{m+n}{2}},$$

for $x \geq -\sqrt{\dfrac{m(n-4)}{2\,(m+n-2)}}$.

The p.d.f. of the standard normal distribution is defined for all x by $f(x) = \frac{1}{\sqrt{2\pi}}\ e^{-x^2/2}$. A graph of the standard normal distribution is the common "bell-shaped curve," which is symmetric about the origin (Figure 5, Section 1.2).

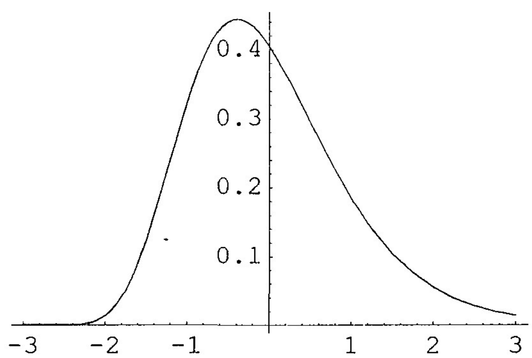

Figure 8: The Standardized $F[50, 50]$.

We now consider the class of functions $\{g_{m,n}(x)\}$ which are the p.d.f.'s of the $F[m, n]$ distributions. Our ultimate goal is to show that $\{g_{m,n}(x)\}$ converges uniformly to $f(x)$ as m and n tend to infinity on any compact interval $[-b, b]$, where $b > 0$. We must note that in order for each function $g_{m,n}(x)$ to be defined for all $x \in [-b, b]$, m and n may initially have to be large enough so that

$$-\sqrt{\frac{m(n-4)}{2\,(m+n-2)}} < -b.$$

2.3 Separating the functions $g_{m,n}(x)$

In order to work with the functions $g_{m,n}(x)$, we shall divide them into three pieces. Thus we rewrite $g_{m,n}(x)$ as follows:

$$g_{m,n}(x) = \sqrt{\frac{2\,n^2(m+n-2)}{m(n-2)^2\,(n-4)}}\;\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\left(\frac{n}{n-2}+\sqrt{\frac{2\,n^2(m+n-2)}{m(n-2)^2\,(n-4)}}\;x\right)^{\frac{m}{2}-1} *$$

$$(\frac{m}{n})^{\frac{m}{2}}\left(1+\frac{m}{n-2}+\frac{m}{n}\sqrt{\frac{2\,n^2(m+n-2)}{m(n-2)^2\,(n-4)}}\;x\right)^{-\frac{m+n}{2}}$$

$$= \frac{n}{n-2}\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\;(\frac{n}{n-2})^{\frac{m}{2}-1}\,(\frac{m+n-2}{n-2})^{-\frac{m}{2}-\frac{n}{2}} *$$

$$(\frac{m}{n})^{\frac{m}{2}}\left(1+\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;x\right)^{\frac{m}{2}-1}\left(1+\sqrt{\frac{2\,m}{(m+n-2)\,(n-4)}}\;x\right)^{-\frac{m+n}{2}}$$

$$= (\frac{m}{m+n-2})^{\frac{m}{2}}\,(\frac{n-2}{m+n-2})^{\frac{n}{2}}\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} *$$

$$\left(1+\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;x\right)^{\frac{m}{2}-1}\left(1+\sqrt{\frac{2\,m}{(m+n-2)\,(n-4)}}\;x\right)^{-\frac{m+n}{2}}.$$

Now we collect the constant terms and write

$$C_{m,n} = (\frac{m}{m+n-2})^{\frac{m}{2}}\,(\frac{n-2}{m+n-2})^{\frac{n}{2}}\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}. \tag{2.3}$$

Next we define $h_{m,n}(x)$ and $k_{m,n}(x)$ by

$$h_{m,n}(x) = \left(1+\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;x\right)^{-1},$$

and

$$k_{m,n}(x) = \left(1 + \sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;x\right)^{\frac{m}{2}} \left(1 + \sqrt{\frac{2\,m}{(m+n-2)\,(n-4)}}\;x\right)^{-\frac{m+n}{2}}.$$

Then $g_{m,n}(x) = C_{m,n}\, h_{m,n}(x)\, k_{m,n}(x)$.

To show that $\{g_{m,n}(x)\}$ converges pointwise to $f(x)$, we will first show the convergence of the separate pieces.

2.4 The limit of the constants

To evaluate the limit of the constants $C_{m,n}$ we shall express the gamma functions in terms of factorials and exponentials. First we shall express $\Gamma(\frac{n}{2})$ in terms of factorials and exponentials in the following lemma.

Lemma 2.1. For positive integers n, $\Gamma(\frac{n}{2})$ can be written as

$$\Gamma(\tfrac{n}{2}) = \begin{cases} \left(\frac{n}{2} - 1\right)!\,, & \text{for } n \text{ even.} \\ \dfrac{(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}\sqrt{\pi}\,, & \text{for } n \text{ odd.} \end{cases}$$

Proof. Since $\Gamma(k) = (k-1)!$ for positive integers k, $\Gamma(\frac{n}{2}) = (\frac{n}{2} - 1)!$ for n even. Now we use the fact that $\Gamma(z) = (z-1)\,\Gamma(z-1)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Next, consider $\Gamma(\frac{n}{2}) = \dfrac{(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}\sqrt{\pi}$ for n odd . We use induction on n, starting with $n = 1$. Clearly this is true when $n = 1$, since

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= \frac{(1-1)!}{2^{1-1}\left(\frac{1-1}{2}\right)!}\sqrt{\pi}.$$

Now we assume

$$\Gamma\left(\frac{k}{2}\right) = \frac{(k-1)!}{2^{k-1}\left(\frac{k-1}{2}\right)!}\sqrt{\pi},$$

for k odd, where $k \in \mathbb{N}$. We must show that

$$\Gamma\left(\frac{k+2}{2}\right) = \frac{((k+2)-1)!}{2^{(k+2)-1}\left(\frac{(k+2)-1}{2}\right)!}\sqrt{\pi},$$

for k odd, where $k \in \mathbb{N}$. We now have

$$\Gamma\left(\frac{k+2}{2}\right) = \left(\frac{k}{2}\right)\Gamma\left(\frac{k}{2}\right)$$

$$= \left(\frac{k}{2}\right)\frac{(k-1)!}{2^{k-1}\left(\frac{k-1}{2}\right)!}\sqrt{\pi}$$

$$= \frac{k(k+1)}{2^2\left(\frac{k+1}{2}\right)}\frac{(k-1)!}{2^{k-1}\left(\frac{k-1}{2}\right)!}\sqrt{\pi}$$

$$= \frac{(k+1)!}{2^{k+1}\left(\frac{k+1}{2}\right)!}\sqrt{\pi}$$

$$= \frac{((k+2)-1)!}{2^{(k+2)-1}\left(\frac{(k+2)-1}{2}\right)!}\sqrt{\pi}.$$

Thus by induction, we have $\Gamma\left(\frac{n}{2}\right) = \frac{(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}\sqrt{\pi}$, for n odd. ■

Likewise we have

$$\Gamma\left(\frac{m}{2}\right) = \begin{cases} \left(\frac{m}{2} - 1\right)! & \text{for } m \text{ even.} \\ \frac{(m-1)!}{2^{m-1}\left(\frac{m-1}{2}\right)!}\sqrt{\pi} & \text{for } m \text{ odd.} \end{cases}$$

Next, we express $\Gamma\left(\frac{m+n}{2}\right)$ in terms of factorials and exponentials.

Lemma 2.2. For positive integers n and m, $\Gamma(\frac{m+n}{2})$ can be written as

$$\Gamma(\frac{m+n}{2}) = \begin{cases} \left(\frac{m+n}{2} - 1\right)!, & \text{for } n \text{ and } m \text{ even; or } n \text{ and } m \text{ odd}. \\ \dfrac{(m+n-1)!}{2^{m+n-1}\left(\frac{m+n-1}{2}\right)!}\sqrt{\pi}, & \text{for } n \text{ odd, } m \text{ even; or } n \text{ even, } m \text{ odd.} \end{cases}$$

Proof. Since $\Gamma(k) = (k-1)!$ for positive integers k, $\Gamma(\frac{m+n}{2}) = \left(\frac{m+n}{2} - 1\right)!$, for m and n even or m and n odd. Now we use the fact that $\Gamma(z) = (z-1)\,\Gamma(z-1)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Next, consider $\Gamma(\frac{m+n}{2}) = \dfrac{(m+n-1)!}{2^{m+n-1}\left(\frac{m+n-1}{2}\right)!}\sqrt{\pi}$ for m even and n odd.

We use induction, starting with $n = 1$ and $m = 2$. Clearly this is true when

$n = 1$and $m = 2$, since

$$\Gamma(\frac{2+1}{2}) = \frac{1}{2}\sqrt{\pi}$$

$$= \frac{(2+1-1)!}{2^{2+1-1}\left(\frac{2+1-1}{2}\right)!}\sqrt{\pi}.$$

Now we assume

$$\Gamma\left(\frac{k+j}{2}\right) = \frac{(k+j-1)!}{2^{k+j-1}\left(\frac{k+j-1}{2}\right)!}\sqrt{\pi},$$

for k even and j odd, where k and j are positive integers. We must show that

$$\Gamma\left(\frac{(k+2)+(j+2)}{2}\right) = \frac{((k+2)+(j+2)-1)!}{2^{(k+2)+(j+2)-1}\left(\frac{(k+2)+(j+2)-1}{2}\right)!}\sqrt{\pi}$$

$$= \frac{(k+j+3)!}{2^{k+j+3}\left(\frac{k+j+3}{2}\right)!}\sqrt{\pi}$$

for k even and j odd, where k and j are positive integers. We now have

$$\Gamma\left(\frac{(k+2)+(j+2)}{2}\right) = \Gamma\left(\frac{k+j}{2}+2\right)$$

$$= \left(\frac{k+j}{2}\right)\left(\frac{k+j}{2}+1\right)\Gamma\left(\frac{k+j}{2}\right)$$

$$= \left(\frac{k+j}{2}\right)\left(\frac{k+j+2}{2}\right)\frac{(k+j-1)!}{2^{k+j-1}\left(\frac{k+j-1}{2}\right)!}\sqrt{\pi}$$

$$= \frac{(k+j+3)(k+j+2)(k+j+1)(k+j)}{2^4\left(\frac{k+j+3}{2}\right)\left(\frac{k+j+1}{2}\right)}\ \frac{(k+j-1)!}{2^{k+j-1}\left(\frac{k+j-1}{2}\right)!}\sqrt{\pi}$$

$$= \frac{(k+j+3)!}{2^{k+j+3}\left(\frac{k+j+3}{2}\right)!}\sqrt{\pi}.$$

Thus by induction, we have $\Gamma(\frac{m+n}{2}) = \frac{(m+n-1)!}{2^{m+n-1}\left(\frac{m+n-1}{2}\right)!}\sqrt{\pi}$, for m even and n odd. Similarly, it can be shown that $\Gamma(\frac{m+n}{2}) = \frac{(m+n-1)!}{2^{m+n-1}\left(\frac{m+n-1}{2}\right)!}\sqrt{\pi}$, for m odd and n even. ■

Now that we have expressed the gamma functions in terms of factorials and exponentials, we substitute these values into Equation (2.3) to obtain the following:

$$C_{m,n} = \left(\frac{m}{m+n-2}\right)^{\frac{m}{2}}\left(\frac{n-2}{m+n-2}\right)^{\frac{n}{2}}\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\ \frac{\left(\frac{m+n}{2}-1\right)!}{\left(\frac{m}{2}-1\right)!\left(\frac{n}{2}-1\right)!},$$

for $m,\ n$ even.

$$C_{m,n} = \left(\frac{m}{m+n-2}\right)^{\frac{m}{2}}\left(\frac{n-2}{m+n-2}\right)^{\frac{n}{2}}\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\ \frac{2^{m+n-2}}{\pi}\ *$$

$$\frac{\left(\frac{m+n}{2}-1\right)!\left(\frac{m-1}{2}\right)!\left(\frac{n-1}{2}\right)!}{(m-1)!\,(n-1)!},$$

for $m,\ n$ odd.

$$C_{m,n} = \left(\frac{m}{m+n-2}\right)^{\frac{m}{2}} \left(\frac{n-2}{m+n-2}\right)^{\frac{n}{2}} \sqrt{\frac{2\,(m+n-2)}{m\;(n-4)}}\; \frac{(m+n-1)!}{2^m\,(n-1)!} *$$

$$\frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{m}{2}-1\right)!\left(\frac{m+n-1}{2}\right)!}, \qquad \text{for } m \text{ even, } n \text{ odd.}$$

$$C_{m,n} = \left(\frac{m}{m+n-2}\right)^{\frac{m}{2}} \left(\frac{n-2}{m+n-2}\right)^{\frac{n}{2}} \sqrt{\frac{2\,(m+n-2)}{m\;(n-4)}}\; \frac{(m+n-1)!}{2^n\,(m-1)!} *$$

$$\frac{\left(\frac{m-1}{2}\right)!}{\left(\frac{n}{2}-1\right)!\left(\frac{m+n-1}{2}\right)!}, \qquad \text{for } m \text{ odd, } n \text{ even.}$$

Now that $C_{m,n}$ has been expressed in terms of factorials and exponentials, we are ready to take the limit of $C_{m,n}$.To evaluate the limit of $\{C_{m,n}\}$, we shall apply Stirling's Formula [7], in the following theorem.

Theorem 2.1. $\lim\limits_{m,n\to\infty} C_{m,n} = \dfrac{1}{\sqrt{2\pi}}$.

Proof. For each case we examine, we will consider the case when m and n tend to infinity at the same rate. (*i*) First we examine the case when m and n are even. In order to evaluate the limit of $C_{m,n}$ when m and n are even, we shall write m in terms of n. Thus we let $m = n + 2c$ for $c \geq -n/2$, where n and c are integers. Then $C_{m,n}$ becomes:

$$C_n = \left(\frac{n+2c}{2\,(n+c-1)}\right)^{\frac{n}{2}+c} \left(\frac{n-2}{2\,(n+c-1)}\right)^{\frac{n}{2}} \sqrt{\frac{4\,(n+c-1)}{(n+2c)\;(n-4)}}\; \frac{(n+c-1)!}{\left(\frac{n+2c}{2}-1\right)!\left(\frac{n}{2}-1\right)!}.$$

Now taking the limit as n tends to infinity we have:

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty}\left(\frac{(n+2c)(n-2)}{4(n+c-1)^2}\right)^{\frac{n}{2}}\left(\frac{n+2c}{2(n+c-1)}\right)^c\sqrt{\frac{4(n+c-1)}{(n+2c)(n-4)}} *$$

$$\frac{\sqrt{2\pi}\,(n+c-1)^{n+c-\frac{1}{2}}\,e^{-(n+c-1)}}{\sqrt{2\pi}\left(\frac{n+2c}{2}-1\right)^{\frac{n}{2}+c-\frac{1}{2}}e^{-\left(\frac{n}{2}+c-1\right)}\sqrt{2\pi}\left(\frac{n}{2}-1\right)^{\frac{n}{2}-\frac{1}{2}}e^{-\left(\frac{n}{2}-1\right)}}$$

$$= \lim_{n\to\infty}\frac{1}{e\sqrt{2\pi}}\left(\frac{(n+2c)(n-2)}{4(n+c-1)^2}\right)^{\frac{n}{2}}\left(\frac{n+2c}{2(n+c-1)}\right)^c\sqrt{\frac{4(n+c-1)}{(n+2c)(n-4)}} *$$

$$\frac{(n+c-1)^{n+c-\frac{1}{2}}}{\left(\frac{n+2c}{2}-1\right)^{\frac{n}{2}+c-\frac{1}{2}}\left(\frac{n}{2}-1\right)^{\frac{n}{2}-\frac{1}{2}}}$$

$$= \frac{1}{e\sqrt{2\pi}}\lim_{n\to\infty}\sqrt{\frac{4(n+c-1)\left(\frac{n+2c}{2}-1\right)\left(\frac{n}{2}-1\right)}{(n+2c)(n-4)(n+c-1)}} *$$

$$\left(\frac{(n+2c)(n+c-1)}{2(n+c-1)\left(\frac{n+2c}{2}-1\right)}\right)^c\left(\frac{(n+2c)(n-2)(n+c-1)^2}{4(n+c-1)^2\left(\frac{n+2c}{2}-1\right)\left(\frac{n}{2}-1\right)}\right)^{\frac{n}{2}}$$

$$= \frac{1}{e\sqrt{2\pi}}\lim_{n\to\infty}\sqrt{\frac{(n+2c-1)(n-1)}{(n+2c)(n-4)}}\left(\frac{n+2c}{n+2c-2}\right)^c\left(\frac{n+2c}{n+2c-2}\right)^{\frac{n}{2}}$$

$$= \frac{1}{e\sqrt{2\pi}}(1)(1)\lim_{n\to\infty}\left(\frac{n+2c}{n+2c-2}\right)^{\frac{n}{2}}.$$

To evaluate the limit of $\left(\frac{n+2c}{n+2c-2}\right)^{\frac{n}{2}}$ as n tends to infinity, it will suffice to evaluate the limit of the natural logarithm. Simplifying the natural logarithm of $\left(\frac{n+2c}{n+2c-2}\right)^{\frac{n}{2}}$ beforehand, we have:

$$\ln\left(\frac{n+2c}{n+2c-2}\right)^{\frac{n}{2}} = \frac{n}{2}\ln\left(\frac{n+2c}{n+2c-2}\right)$$

$$= \frac{\ln\left(\frac{n+2c}{n+2c-2}\right)}{\frac{2}{n}}.$$

Now taking the limit as n tends to infinity, we have the indeterminate expression 0/0; thus we apply L'Hopital's Rule. Taking the derivatives of both the numerator and the denominator with respect to n, we obtain:

$$\frac{\frac{d}{dn}\left[\ln\left(\frac{n+2c}{n+2c-2}\right)\right]}{\frac{d}{dn}\left[\frac{2}{n}\right]} = \frac{n^2}{(n+2c)(n+2c-2)}.$$

The limit as n tends to infinity is now the indeterminate expression ∞/∞; but two more applications of L'Hopital's Rule yield a limit of 1. Since we had taken the natural logarithm of $\left(\frac{n+2c}{n+2c-2}\right)^{\frac{n}{2}}$, we find that

$$\lim_{n\to\infty}\left(\frac{n+2c}{n+2c-2}\right)^{\frac{n}{2}} = e\,.$$

Thus, when m and n are even:

$$\lim_{n\to\infty} C_n = \frac{1}{e\sqrt{2\pi}}\,(1)\,(1)\,e = \frac{1}{\sqrt{2\pi}}.$$

(ii) Next we examine the case when m and n are odd. To evaluate the limit of $C_{m,n}$ when m and n are odd, we write m in terms of n. We again let $m = n + 2c$ for $c \geq -n/2$, where n and c are integers. Then $C_{m,n}$ becomes:

$$C_n = \left(\frac{n+2c}{2(n+c-1)}\right)^{\frac{n}{2}+c}\left(\frac{n-2}{2(n+c-1)}\right)^{\frac{n}{2}}\sqrt{\frac{4(n+c-1)}{(n+2c)(n-4)}}\;\frac{2^{2n+2c-2}}{\pi}\;*$$

$$\frac{(n+c-1)!\left(\frac{n+2c-1}{2}\right)!\left(\frac{n-1}{2}\right)!}{(n+2c-1)!\,(n-1)!}.$$

Now taking the limit as n tends to infinity we have:

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty}\left(\frac{n+2c}{2(n+c-1)}\right)^{\frac{n}{2}+c}\left(\frac{n-2}{2(n+c-1)}\right)^{c}\sqrt{\frac{4(n+c-1)}{(n+2c)(n-4)}}\ *$$

$$\frac{2^{2n+2c-2}\sqrt{2\pi}\,(n+c-1)^{n+c-\frac{1}{2}}\,e^{-(n+c-1)}\sqrt{2\pi}\left(\frac{n+2c-1}{2}\right)^{\frac{n}{2}+c}}{\pi\sqrt{2\pi}\,(n+2c-1)^{n+2c-\frac{1}{2}}\,e^{-(n+2c-1)}}\ *$$

$$\frac{e^{-\left(\frac{n}{2}+c-\frac{1}{2}\right)}\sqrt{2\pi}\left(\frac{n-1}{2}\right)^{\frac{n}{2}}e^{-\left(\frac{n}{2}-\frac{1}{2}\right)}}{\sqrt{2\pi}\,(n-1)^{n-\frac{1}{2}}\,e^{-(n-1)}}$$

$$= \lim_{n\to\infty}\left(\frac{(n+2c)(n-2)}{4(n+c-1)^2}\right)^{\frac{n}{2}}\left(\frac{n+2c}{2(n+c-1)}\right)^{c}\sqrt{\frac{n+c-1}{(n+2c)(n-4)}}\ \frac{2^{2n+2c}}{\sqrt{2\pi}}\ *$$

$$\frac{(n+c-1)^{n+c-\frac{1}{2}}\left(\frac{n+2c-1}{2}\right)^{\frac{n}{2}+c}\left(\frac{n-1}{2}\right)^{\frac{n}{2}}}{(n+2c-1)^{n+2c-\frac{1}{2}}\,(n-1)^{n-\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{n\to\infty}\left(\frac{2^4(n+2c)(n-2)(n+c-1)^2\left(\frac{n+2c-1}{2}\right)\left(\frac{n-1}{2}\right)}{4(n+c-1)^2(n+2c-1)^2(n-1)^2}\right)^{\frac{n}{2}}\ *$$

$$\sqrt{\frac{(n+c-1)(n+2c-1)(n-1)}{(n+2c)(n-4)(n+c-1)}}\left(\frac{2^2(n+2c)(n+c-1)\left(\frac{n+2c-1}{2}\right)}{2(n+c-1)(n+2c-1)^2}\right)^{c}$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{n\to\infty}\sqrt{\frac{(n+2c-1)(n-1)}{(n+2c)(n-4)}}\left(\frac{n+2c}{n+2c-1}\right)^{c}\left(\frac{(n+2c)(n-2)}{(n+2c-1)(n-1)}\right)^{\frac{n}{2}}$$

$$= \frac{1}{\sqrt{2\pi}}(1)(1)\lim_{n\to\infty}\left(\frac{(n+2c)(n-2)}{(n+2c-1)(n-1)}\right)^{\frac{n}{2}}.$$

In order to evaluate the limit of $\left(\frac{(n+2c)(n-2)}{(n+2c-1)(n-1)}\right)^{\frac{n}{2}}$ as n tends to infinity, it will suffice to evaluate the limit of the natural logarithm. Simplifying the natural logarithm of $\left(\frac{(n+2c)(n-2)}{(n+2c-1)(n-1)}\right)^{\frac{n}{2}}$ beforehand, we have:

$$\ln\left(\frac{(n+2\,c)\,(n-2)}{(n+2\,c-1)\,(n-1)}\right)^{\frac{n}{2}} = \frac{n}{2}\ln\left(\frac{(n+2\,c)\,(n-2)}{(n+2\,c-1)\,(n-1)}\right)$$

$$= \frac{\ln\left(\frac{(n+2\,c)\,(n-2)}{(n+2\,c-1)\,(n-1)}\right)}{\frac{2}{n}}\,.$$

Now taking the limit as n tends to infinity, we have the indeterminate expression 0/0; thus we apply L'Hopital's Rule. Taking the derivatives of both the numerator and the denominator with respect to n, we obtain:

$$\frac{\frac{d}{dn}\left[\ln\left(\frac{(n+2\,c)\,(n-2)}{(n+2\,c-1)\,(n-1)}\right)\right]}{\frac{d}{dn}\left[\frac{2}{n}\right]} = \frac{-n^2(n+c-1)\,(2\,c+1)}{(n-2)\,(n-1)\,(n+2\,c)\,(n+2\,c-1)}\,.$$

The limit as n tends to infinity is now the indeterminate expression ∞/∞; but three more applications of L'Hopital's Rule yield a limit of 0. Since we had taken the natural logarithm of $\left(\frac{(n+2\,c)\,(n-2)}{(n+2\,c-1)\,(n-1)}\right)^{\frac{n}{2}}$, we find that

$$\lim_{n\to\infty}\left(\frac{(n+2\,c)\,(n-2)}{(n+2\,c-1)\,(n-1)}\right)^{\frac{n}{2}} = e^0 = 1\,.$$

Thus, when m and n are odd:

$$\lim_{n\to\infty} C_n = \frac{1}{\sqrt{2\,\pi}}\,(1)\,(1)\,(1) = \frac{1}{\sqrt{2\,\pi}}\,.$$

(*iii*) Finally we examine the case when m is even and n is odd. In order to evaluate the limit of $C_{m,\,n}$ when m is even and n is odd, we write m in terms of n. Thus we let $m = n + 2\,c + 1$ for $c \geq -(n+1)/2$, where n and c are integers. Then $C_{m,\,n}$ becomes:

$$C_n = \left(\frac{n+2c+1}{2n+2c-1}\right)^{\frac{n}{2}+c+\frac{1}{2}} \left(\frac{n-2}{2n+2c-1}\right)^{\frac{n}{2}} \sqrt{\frac{2(2n+2c-1)}{(n+2c+1)(n-4)}} *$$

$$\frac{(2n+2c)!\left(\frac{n-1}{2}\right)!}{2^{n+2c+1}(n-1)!\left(\frac{n+2c-1}{2}\right)!(n+c)!}.$$

Now taking the limit as n tends to infinity we have:

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \left(\frac{n+2c+1}{2n+2c-1}\right)^{\frac{n}{2}+c+\frac{1}{2}} \left(\frac{n-2}{2n+2c-1}\right)^{\frac{n}{2}} \sqrt{\frac{2(2n+2c-1)}{(n+2c+1)(n-4)}} *$$

$$\frac{\sqrt{2\pi}\,(2n+2c)^{2n+2c+\frac{1}{2}}\, e^{-(2n+2c)}}{\sqrt{2\pi}\,(n-1)^{n-\frac{1}{2}}\, e^{-(n-1)}\sqrt{2\pi}\left(\frac{n+2c-1}{2}\right)^{\frac{n}{2}+c} e^{-\left(\frac{n}{2}+c-\frac{1}{2}\right)}} *$$

$$\frac{\sqrt{2\pi}\left(\frac{n-1}{2}\right)^{\frac{n}{2}} e^{-\left(\frac{n}{2}-\frac{1}{2}\right)}}{2^{n+2c+1}\sqrt{2\pi}\,(n+c)^{n+c+\frac{1}{2}}\, e^{-(n+c)}}$$

$$= \lim_{n\to\infty} \left(\frac{n+2c+1}{2n+2c-1}\right)^{\frac{n}{2}+c+\frac{1}{2}} \left(\frac{n-2}{2n+2c-1}\right)^{\frac{n}{2}} \sqrt{\frac{2(2n+2c-1)}{(n+2c+1)(n-4)}} *$$

$$\frac{(2n+2c)^{2n+2c+\frac{1}{2}}\left(\frac{n-1}{2}\right)^{\frac{n}{2}}}{2^{n+2c+1}\, e\sqrt{2\pi}\,(n-1)^{n-\frac{1}{2}}\left(\frac{n+2c-1}{2}\right)^{\frac{n}{2}+c}(n+c)^{n+c+\frac{1}{2}}}$$

$$= \frac{1}{e\sqrt{2\pi}} \lim_{n\to\infty} \sqrt{\frac{n-1}{n-4}} \left(\frac{(n+2c+1)(2n+2c)^2}{2^2(2n+2c-1)\left(\frac{n+2c-1}{2}\right)(n+c)}\right)^c *$$

$$\left(\frac{(n+2c+1)(n-2)(2n+2c)^4\left(\frac{n-1}{2}\right)}{2^2(2n+2c-1)(2n+2c-1)(n-1)^2\left(\frac{n+2c-1}{2}\right)(n+c)^2}\right)^{\frac{n}{2}}$$

$$= \frac{1}{e\sqrt{2\pi}} \lim_{n\to\infty} \sqrt{\frac{n-1}{n-4}} \left(\frac{2(n+2c+1)(n+c)}{(2n+2c-1)(n+2c-1)}\right)^c *$$

$$\left(\frac{2^2(n+2c+1)(n-2)(n+c)^2}{(2n+2c-1)^2(n-1)(n+2c-1)}\right)^{\frac{n}{2}}$$

$$= \frac{1}{e\sqrt{2\pi}}(1)(1)\lim_{n\to\infty}\left(\frac{(n+2c+1)(n-2)}{(n-1)(n+2c-1)}\right)^{\frac{n}{2}}\left(\frac{n+c}{n+c-\frac{1}{2}}\right)^{n}.$$

To complete the evaluation of this limit as n tends to infinity, it will suffice to evaluate the limit of the natural logarithm of $\left(\frac{(n+2c+1)(n-2)}{(n-1)(n+2c-1)}\right)^{\frac{n}{2}}$ and $\left(\frac{n+c}{n+c-\frac{1}{2}}\right)^{n}$. First we shall examine the limit of $\left(\frac{(n+2c+1)(n-2)}{(n-1)(n+2c-1)}\right)^{\frac{n}{2}}$. Simplifying the natural logarithm beforehand, we have:

$$\ln\left(\frac{(n+2c+1)(n-2)}{(n-1)(n+2c-1)}\right)^{\frac{n}{2}} = \frac{\ln\left(\frac{(n+2c+1)(n-2)}{(n-1)(n+2c-1)}\right)}{\frac{2}{n}}.$$

Now taking the limit as n tends to infinity, we have the indeterminate expression 0/0. Thus we apply L'Hopital's Rule to obtain:

$$\frac{\frac{d}{dn}\left[\ln\left(\frac{(n+2c+1)(n-2)}{(n-1)(n+2c-1)}\right)\right]}{\frac{d}{dn}\left[\frac{2}{n}\right]} = \frac{n^2(n^2-4cn-6n-4c^2+5)}{2(n-1)(n-2)(n+2c+1)(n+2c-1)}.$$

As n tends to infinity, the limit is now the indeterminate expression ∞/∞; however, repeated applications of L'Hopital's Rule yield a limit of $\frac{1}{2}$. Since we had taken the natural logarithm of $\left(\frac{(n+2c+1)(n-2)}{(n-1)(n+2c-1)}\right)^{\frac{n}{2}}$, we find that

$$\lim_{n\to\infty}\left(\frac{(n+2c+1)(n-2)}{(n-1)(n+2c-1)}\right)^{\frac{n}{2}} = e^{\frac{1}{2}}.$$

Next we shall examine the limit of $\left(\frac{n+c}{n+c-\frac{1}{2}}\right)^{n}$. Simplifying the natural logarithm beforehand, we have:

$$\ln\left(\frac{n+c}{n+c-\frac{1}{2}}\right)^n = \frac{\ln\left(\frac{n+c}{n+c-\frac{1}{2}}\right)}{\frac{1}{n}}.$$

Taking the limit as n tends to infinity, we have the indeterminate expression 0/0. Thus we apply L'Hopital's Rule to obtain:

$$\frac{\frac{d}{dn}\left[\ln\left(\frac{n+c}{n+c-\frac{1}{2}}\right)\right]}{\frac{d}{dn}\left[\frac{1}{n}\right]} = \frac{n^2}{(2\,n+2\,c-1)\,(n+c)}.$$

As n tends to infinity, the limit is now the indeterminate expression ∞/∞. Two more applications of L'Hopital's Rule yield a limit of $\frac{1}{2}$. Since we had taken the natural logarithm of $\ln\left(\frac{n+c}{n+c-\frac{1}{2}}\right)^n$, we find that

$$\lim_{n\to\infty} \ln\left(\frac{n+c}{n+c-\frac{1}{2}}\right)^n = e^{\frac{1}{2}}.$$

Thus, when m is even and n is odd:

$$\lim_{n\to\infty} C_n = \frac{1}{e\sqrt{2\pi}}\,(1)\,(1)\,e^{\frac{1}{2}}\,e^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}.$$

Similarly, it can be shown that when m is odd and n is even, the limit of $C_{m,n}$ is $\frac{1}{\sqrt{2\pi}}$. Since the limit of $C_{m,n}$ as m and n tend to infinity is C, the sequence $\{C_{m,n}\}$ converges pointwise to C. ■

2.5 Pointwise convergence of $h_{m,n}(x)\,k_{m,n}(x)$

We shall now show that $h_{m,n}(x)\,k_{m,n}(x)$ converges pointwise to $g(x)$ by first showing the convergence of the separate pieces $h_{m,n}(x)$ and $k_{m,n}(x)$.

Theorem 2.2. For any interval $[-b, \infty)$, $\lim_{m,n\to\infty} h_{m,n}(x) = 1$.

Proof. Let $h_{m,n}(x) = \dfrac{1}{1+\sqrt{\dfrac{2(m+n-2)}{m(n-4)}}\,x}$. Thus there are singularities at $x = -\sqrt{\dfrac{m(n-4)}{2(m+n-2)}}$. However, the singularities are avoided on the interval $[-b, \infty)$ by initially choosing m and n large enough. Now to evaluate the limit of $h_{m,n}(x)$ as m and n tend to infinity at the same rate, we write m in terms of n. Thus we let $m = n + p$ for $p \geq -n$, where p is an integer. Then $h_{m,n}(x)$ becomes :

$$h_n(x) = \frac{1}{1+\sqrt{\dfrac{2(2n+p-2)}{(n+p)(n-4)}}\,x}.$$

Now taking the limit as n tends to infinity we have:

$$\begin{aligned}\lim_{n\to\infty} h_n(x) &= \lim_{n\to\infty} \frac{1}{1+\sqrt{\dfrac{2(2n+p-2)}{(n+p)(n-4)}}\,x}\\[2ex] &= \lim_{n\to\infty} \frac{1}{1+\dfrac{2}{\sqrt{n}}x}\\[2ex] &= 1.\end{aligned}$$

Thus $\{h_{m,n}(x)\}$ converges pointwise to 1 on the interval $[-b, \infty)$. ∎

We next show that $\lim_{m,n\to\infty} k_{m,n}(x) = e^{-x^2/2}$.

Theorem 2.3. For any interval $[-b, \infty)$, $\lim_{m,n\to\infty} k_{m,n}(x) = e^{-x^2/2}$.

Proof. Let $k_{m,n}(x) = \left(1 + \sqrt{\dfrac{2(m+n-2)}{m(n-4)}}\;x\right)^{\frac{m}{2}} \left(1 + \sqrt{\dfrac{2m}{(m+n-2)(n-4)}}\;x\right)^{-\frac{m+n}{2}}$.

In order for the functions to be defined for x, we must let m and n be large enough so that $-\sqrt{\frac{m(n-4)}{2\,(m+n-2)}} < -b$. To evaluate the limit of $k_{m,\,n}(x)$ as m and n tend to infinity at the same rate, we write m in terms of n. Thus we let $m = n + p$ for $p \geq -n$. Then $k_{m,\,n}(x)$ becomes

$$k_n(x) = \left(1 + \sqrt{\frac{2\,(2\,n+p-2)}{(n+p)\,(n-4)}}\;x\right)^{\frac{n+p}{2}} \left(1 + \sqrt{\frac{2\,(n+p)}{(2\,n+p-2)\,(n-4)}}\;x\right)^{-n-\frac{p}{2}}.$$

Now taking the limit as n tends to infinity we have:

$$\lim_{n\to\infty} k_n(x) = \lim_{n\to\infty} \left(1 + \sqrt{\frac{2\,(2\,n+p-2)}{(n+p)\,(n-4)}}\;x\right)^{\frac{n}{2}} \left(1 + \sqrt{\frac{2\,(2\,n+p-2)}{(n+p)\,(n-4)}}\;x\right)^{\frac{p}{2}} *$$

$$\left(1 + \sqrt{\frac{2\,(n+p)}{(2\,n+p-2)\,(n-4)}}\;x\right)^{-n} \left(1 + \sqrt{\frac{2\,(n+p)}{(2\,n+p-2)\,(n-4)}}\;x\right)^{-\frac{p}{2}}$$

$$= \lim_{n\to\infty} \left(1 + \frac{2}{\sqrt{n}}\;x\right)^{\frac{p}{2}} \left(1 + \frac{1}{\sqrt{n}}\;x\right)^{-\frac{p}{2}} \left(1 + \frac{2}{\sqrt{n}}\;x\right)^{\frac{n}{2}} *$$

$$\left(1 + \frac{1}{\sqrt{n}}\;x\right)^{-n}$$

$$= \lim_{n\to\infty} \left(1 + \frac{2}{\sqrt{n}}\;x\right)^{\frac{n}{2}} \left(1 + \frac{1}{\sqrt{n}}\;x\right)^{-n}.$$

In order to complete the evaluation of this limit as n tends to infinity, it suffices to evaluate the limit of the natural logarithm of $\left(1 + \frac{2}{\sqrt{n}}\;x\right)^{\frac{n}{2}} \left(1 + \frac{1}{\sqrt{n}}\,x\right)^{-n}$. Simplifying the natural logarithm beforehand, we have:

$$\ln\left(\left(1 + \frac{2}{\sqrt{n}}\;x\right)^{\frac{n}{2}} \left(1 + \frac{1}{\sqrt{n}}\;x\right)^{-n}\right) = \frac{\ln\left(1+\frac{2\,x}{\sqrt{n}}\right) - 2\ln\left(1+\frac{1\,x}{\sqrt{n}}\right)}{\frac{2}{n}}.$$

Now taking the limit as n tends to infinity, we have the indeterminate expression 0/0. Thus we apply L'Hopital's Rule to obtain:

$$\frac{\frac{d}{dn}\left[\ln\left(1+\frac{2x}{\sqrt{n}}\right)-2\ln\left(1+\frac{1x}{\sqrt{n}}\right)\right]}{\frac{d}{dn}\left[\frac{2}{n}\right]} = \frac{-n^2}{2}\left(\frac{-x}{n\left(\sqrt{n}+2x\right)}+\frac{x}{n\left(\sqrt{n}+x\right)}\right)$$

$$= \frac{-n^2}{2}\left(\frac{n x^2+n^{3/2} x-2 n x^2-n^{3/2} x}{n^2\left(\sqrt{n}+2x\right)\left(\sqrt{n}+x\right)}\right)$$

$$= \frac{-1}{2}\left(\frac{n x^2}{\left(\sqrt{n}+2x\right)\left(\sqrt{n}+x\right)}\right)$$

$$= \frac{-x^2}{2}\left(\frac{1}{\left(1+\frac{2x}{\sqrt{n}}\right)}\right)\left(\frac{1}{\left(1+\frac{x}{\sqrt{n}}\right)}\right).$$

Now taking the limit as n tends to infinity we have:

$$\lim_{n\to\infty}\frac{-x^2}{2}\left(\frac{1}{\left(1+\frac{2x}{\sqrt{n}}\right)}\right)\left(\frac{1}{\left(1+\frac{x}{\sqrt{n}}\right)}\right) = \frac{-x^2}{2}.$$

Since we had taken the natural logarithm of $\left(1+\frac{2}{\sqrt{n}}\,x\right)^{\frac{n}{2}}\left(1+\frac{1}{\sqrt{n}}\,x\right)^{-n}$, we find that

$$\lim_{n\to\infty}\left(1+\frac{2}{\sqrt{n}}\,x\right)^{\frac{n}{2}}\left(1+\frac{1}{\sqrt{n}}\,x\right)^{-n} = e^{-\frac{x^2}{2}}.$$

Therefore $\{k_{m,n}(x)\}$ converges pointwise to $e^{-\frac{x^2}{2}}$ on the interval $[-b, \infty)$. ■

We have shown that the separate pieces of $\{g_{m,n}(x)\}$ converge pointwise to their respective limits on $[-b, \infty)$. Thus the product still converges pointwise, which allows us to state the following result.

Corollary 2.1. For any x in the interval $[-b, \infty)$,

$$\lim_{m,n\to\infty} \left(\frac{m}{m+n-2}\right)^{\frac{m}{2}} \left(\frac{n-2}{m+n-2}\right)^{\frac{n}{2}} \sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\ \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} *$$

$$\left(1+\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\ x\right)^{\frac{m}{2}-1} \left(1+\sqrt{\frac{2\,m}{(m+n-2)\,(n-4)}}\ x\right)^{-\frac{m+n}{2}} = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}.$$

2.6 Uniform convergence of the F-Distribution

We will now show that on $[-b, b]$, the sequence of functions $\{h_{m,n}(x)\}$ converges uniformly to 1, the sequence of functions $\{k_{m,n}(x)\}$ converges uniformly to $e^{-x^2/2}$, and the product $\{h_{m,n}(x)\, k_{m,n}(x)\}$ converges uniformly to $e^{-x^2/2}$. We will then conclude this chapter by proving that the sequence of functions $\{g_{m,n}(x)\} = \{C_{m,n}\, h_{m,n}(x)\, k_{m,n}(x)\}$ converges uniformly to $\frac{1}{\sqrt{2\pi}}\, e^{\frac{-x^2}{2}}$ on $[-b, b]$.

We have seen that the separate pieces of $\{g_{m,n}(x)\}$ converges pointwise to their respective limits. As stated in the introduction, to show that a sequence of functions converges uniformly to $f(x)$, we must show that for each $\varepsilon > 0$ there exists an integer N such that for $m,\ n \geq N$, $f_{m,n}(x)$ is within ε of $f(x)$ for all $x \in [-b, b]$ [1]. For the sequence of functions $\{h_{m,n}(x)\}$ this result is easily shown in the following lemma.

Lemma 2.3. On any interval $[-b, b]$, the sequence of functions $\{h_{m,n}(x)\}$

$= \left\{ \dfrac{1}{1+\sqrt{\dfrac{2(m+n-2)}{m(n-4)}}\, x} \right\}$ converges uniformly to 1, when $m = n$.

Proof. Let $m = n$. Then $h_{m,n}(x)$ becomes:

$$h_n(x) = \frac{1}{1+2x\sqrt{\dfrac{n-1}{n(n-4)}}}.$$

Now let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \dfrac{1}{1+2b\sqrt{\dfrac{n-1}{n(n-4)}}} = \lim\limits_{n\to\infty} \dfrac{1}{1+\dfrac{2b}{\sqrt{n}}} = 1$ and

$\lim\limits_{n\to\infty} \dfrac{1}{1-2b\sqrt{\dfrac{n-1}{n(n-4)}}} = \lim\limits_{n\to\infty} \dfrac{1}{1-\dfrac{2b}{\sqrt{n}}} = 1$, there exists and integer N such that

if $m, n \geq N$, then $-\frac{1}{2}\cdot\sqrt{\dfrac{n(n-4)}{n-1}} < -b$, $\left| \dfrac{1}{1+2b\sqrt{\dfrac{n-1}{n(n-4)}}} - 1 \right| < \varepsilon$, and

$\left| \dfrac{1}{1-2b\sqrt{\dfrac{n-1}{n(n-4)}}} - 1 \right| < \varepsilon$. Then for all $n \geq N$,

$$1 - \varepsilon < \frac{1}{1+2b\sqrt{\dfrac{n-1}{n(n-4)}}} \leq \frac{1}{1+2x\sqrt{\dfrac{n-1}{n(n-4)}}} \leq 1,$$

for $0 \leq x \leq b$; and

$$1 \leq \frac{1}{1+2x\sqrt{\dfrac{n-1}{n(n-4)}}} \leq \frac{1}{1-2b\sqrt{\dfrac{n-1}{n(n-4)}}} < 1 + \varepsilon,$$

for $-b \leq x \leq 0$. Thus for all $x \in [-b, b]$, $\dfrac{1}{1+2x\sqrt{\dfrac{n-1}{n(n-4)}}}$ is within ε of 1 if

$n \geq N$. Therefore the sequence of functions $\{h_{m,n}(x)\}$ converges uniformly to 1.

Now we consider the sequence of functions $\{k_{m,n}(x)\}$. Before we show the uniform convergence of $\{k_{m,n}(x)\}$, we shall first show that two sequences of functions bound $\{k_{m,n}(x)\}$ when $m = n$, in the following proposition.

Proposition 2.1. If $m = n$, $\{T_n(x)\} = \left(1 + \frac{2x}{\sqrt{n+4}}\right)^{\frac{n}{2}} \left(1 + \frac{x}{\sqrt{n+4}}\right)^{-(n+4)}$, and $\{U_n(x)\} = \left(1 + \frac{2x}{\sqrt{n-4}}\right)^{\frac{n}{2}} \left(1 + \frac{x}{\sqrt{n-4}}\right)^{-n}$, then $T_n(x) \le k_n(x) \le U_n(x)$ for $x \ge 0$, and $U_n(x) \le k_n(x) \le T_n(x)$, for $x \le 0$.

Proof. Let $m = n$, $\{T_n(x)\} = \left(1 + \frac{2x}{\sqrt{n+4}}\right)^{\frac{n}{2}} \left(1 + \frac{x}{\sqrt{n+4}}\right)^{-(n+4)}$, and $\{U_n(x)\} = \left(1 + \frac{2x}{\sqrt{n-4}}\right)^{\frac{n}{2}} \left(1 + \frac{x}{\sqrt{n-4}}\right)^{-n}$. Then $k_{m,n}(x)$ becomes:

$$k_n(x) = \left(1 + 2x\sqrt{\frac{n-1}{n\,(n-4)}}\right)^{\frac{n}{2}} \left(1 + x\sqrt{\frac{n}{(n-1)\,(n-4)}}\right)^{-n}.$$

First we consider the relationship between the sequences of functions $\{U_n(x)\}$ and $\{k_n(x)\}$. For $x \ge 0$,

$$k_n(x) = \frac{\left(1+2x\sqrt{\frac{n-1}{n\,(n-4)}}\right)^{\frac{n}{2}}}{\left(1+x\sqrt{\frac{n}{(n-1)\,(n-4)}}\right)^{n}} \le \frac{\left(1+2x\sqrt{\frac{n}{n\,(n-4)}}\right)^{\frac{n}{2}}}{\left(1+x\sqrt{\frac{n-1}{(n-1)\,(n-4)}}\right)^{n}}$$

$$= \frac{\left(1+\frac{2x}{\sqrt{(n-4)}}\right)^{\frac{n}{2}}}{\left(1+\frac{x}{\sqrt{(n-4)}}\right)^{n}}$$

$$= U_n(x).$$

For $x \leq 0$, the inequality is reversed. Figures 9 and 10 demonstrate the relationship between $U_n(x)$ and $k_n(x)$.

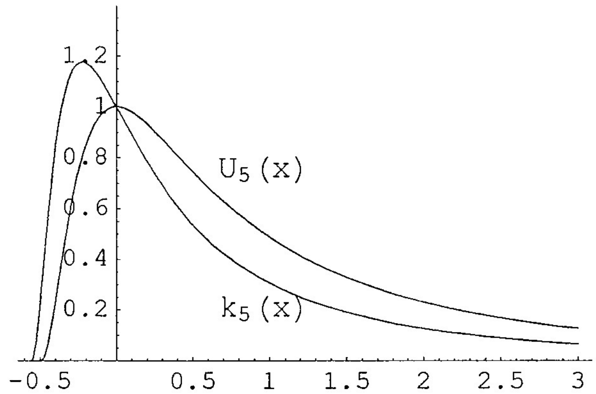

Figure 9: $k_n(x)$ and $U_n(x)$, when $n = 5$.

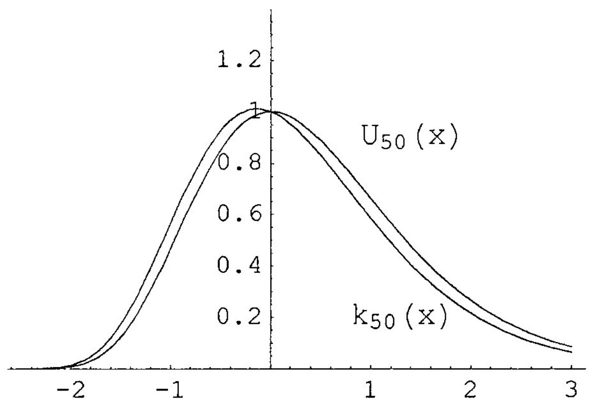

Figure 10: $k_n(x)$ and $U_n(x)$, when $n = 50$.

Next we shall employ the use of *Mathematica* to examine the relationship between the sequences of functions $\{T_n(x)\}$ and $\{k_n(x)\}$. We load the following *Mathematica* package, which provides a means for solving inequalities.

```
<< Algebra`InequalitySolve`
```

The *Mathematica* command `InequalitySolve[`*expr*`, `*x*`]` finds conditions that must be satisfied by real values of x in order for the expression *expr* to be true. To use this command we shall formulate a table that will assign values to n and find the conditions necessary for the expression *expr* to be true. For $k_n(x) \geq T_n(x)$ we have

```
f = InequalitySolve[
```

$$\left(1+\sqrt{\frac{4*(n-1)}{n*(n-4)}}\,x\right)^{\frac{n}{2}} * \left(1+\sqrt{\frac{n}{(n-1)*(n-4)}}\,x\right)^{-n} \geq \left(1+\frac{1}{\sqrt{n+4}}\,x\right)^{n} * \left(1+\frac{1}{\sqrt{n+4}}\,x\right)^{-(n+4)}, x];$$

```
TableForm[Table[{n, f}, {n, 5, 5000, 500}],
 TableHeadings → {Automatic, {"n", "x"}}]
```

	n	x
1	5	x ≥ 0
2	505	x ≥ 0
3	1005	x ≥ 0
4	1505	x ≥ 0
5	2005	x ≥ 0
6	2505	x ≥ 0
7	3005	x ≥ 0
8	3505	x ≥ 0
9	4005	x ≥ 0
10	4505	x ≥ 0,

and for $k_n(x) \leq T_n(x)$ we have

```
f = InequalitySolve[
```

$$\left(1+\sqrt{\frac{4*(n-1)}{n*(n-4)}}\,x\right)^{\frac{n}{2}} * \left(1+\sqrt{\frac{n}{(n-1)*(n-4)}}\,x\right)^{-n} \leq \left(1+\frac{1}{\sqrt{n+4}}\,x\right)^{n} * \left(1+\frac{1}{\sqrt{n+4}}\,x\right)^{-(n+4)}, x];$$

```
TableForm[Table[{n, f}, {n, 5, 5000, 500}],
 TableHeadings → {Automatic, {"n", "x"}}]
```

	n	x
1	5	x ≤ 0
2	505	x ≤ 0
3	1005	x ≤ 0
4	1505	x ≤ 0
5	2005	x ≤ 0
6	2505	x ≤ 0
7	3005	x ≤ 0
8	3505	x ≤ 0
9	4005	x ≤ 0
10	4505	x ≤ 0.

The two tables show numerically that as the value of n increases, $k_n(x) \geq T_n(x)$ for $x \geq 0$ and $k_n(x) \leq T_n(x)$ for $x \leq 0$. We can see graphically that this relationship holds in Figures 11, 12 and 13.

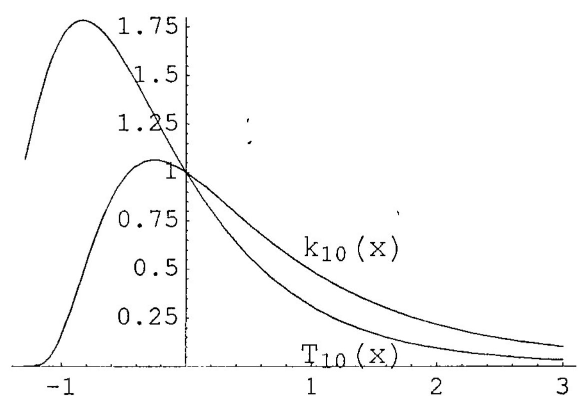

Figure 11: $k_n(x)$ and $T_n(x)$, when $n = 10$.

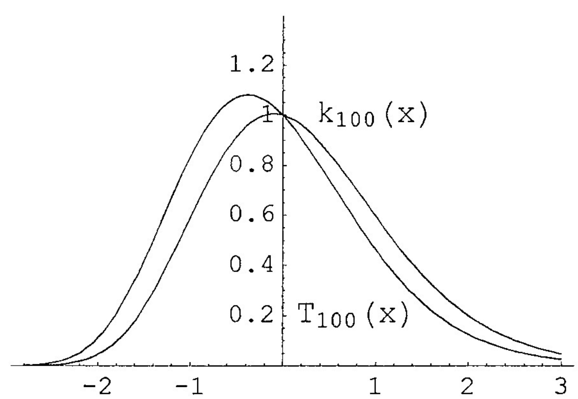

Figure 12: $k_n(x)$ and $T_n(x)$, when $n = 100$.

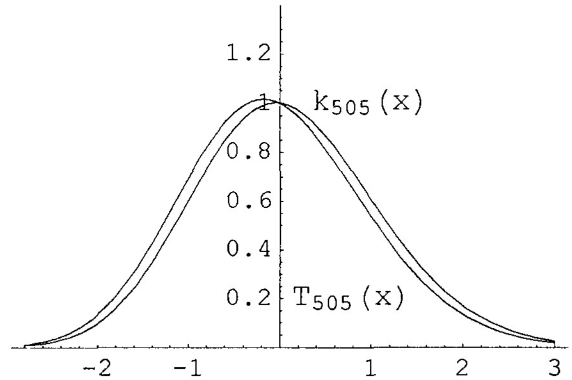

Figure 13: $k_n(x)$ and $T_n(x)$, when $n = 505$.

The numerical and graphical evidence supports the assumption that $k_n(x) \geq T_n(x)$ for $x \geq 0$ and $k_n(x) \leq T_n(x)$ for $x \leq 0$; however, the analytical proof that this relationship holds eludes the author at this time. Henceforth we will assume that the relationship between $\{T_n\}$ and $\{k_n(x)\}$ can be proven analytically. Thus $T_n(x) \leq k_n(x) \leq U_n(x)$ for $x \geq 0$, and $U_n(x) \leq k_n(x) \leq T_n(x)$, for $x \leq 0$. ∎

Figure 14 demonstrates the relationship between $T_n(x)$, $k_n(x)$ and $U_n(x)$.

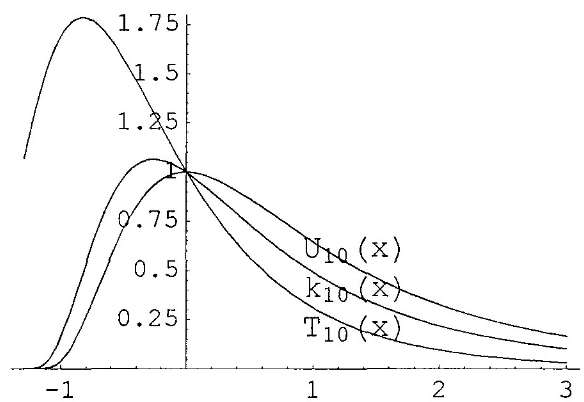

Figure 14: $T_n(x)$, $k_n(x)$ and $U_n(x)$, when $n = 10$.

We now show that the two sequences of functions that bound $\{k_n(x)\}$ are uniformly convergent. First we consider the sequence of functions $\{U_n(x)\}$.

Lemma 2.4. On any interval $[-b, b]$, the sequence of functions $U_n(x) = \left(1 + \frac{2x}{\sqrt{n-4}}\right)^{\frac{n}{2}} \left(1 + \frac{x}{\sqrt{n-4}}\right)^{-n}$ converges uniformly to $e^{-x^2/2}$.

Proof. First we show that $U_n(x)$ converges pointwise to $e^{-x^2/2}$. For any interval $[-b, \infty)$,

$$\begin{aligned}
\lim_{n\to\infty} U_n(x) &= \lim_{n\to\infty} \left(1 + \frac{2x}{\sqrt{n-4}}\right)^{\frac{n}{2}} \left(1 + \frac{x}{\sqrt{n-4}}\right)^{-n} \\
&= \lim_{n\to\infty} \left(1 + \frac{2}{\sqrt{n}}\ x\right)^{\frac{n}{2}} \left(1 + \frac{1}{\sqrt{n}}\ x\right)^{-n} \\
&= e^{-\frac{x^2}{2}}.
\end{aligned}$$

[See argument in Theorem 2.3]. Now by letting $v = \sqrt{n-4}$, it will suffice to show that the sequence of functions $\{U_v(x)\} = \left\{\left(1 + \frac{2x}{v}\right)^{\frac{v^2+4}{2}} (1 + \frac{x}{v})^{-(v^2+4)}\right\}$ converges uniformly to $e^{-x^2/2}$, as v tends to infinity. We initially start our sequences with n large enough so that $-\sqrt{n-4} < -b$. Thus for our new sequence written in terms of v, we have that $v > b$. Thus for all $[-b, b]$, we have that $-1 < \frac{x}{v} < 1$. It is also the case that this new sequence still converges pointwise to $e^{-x^2/2}$.

To show that the sequence of functions $\{U_v(x)\}$ converges uniformly to $e^{-x^2/2}$, we shall divide $\{U_v(x)\}$ into two pieces. Thus we rewrite $\{U_v(x)\}$ as follows:

$$U_v(x) = \left(1 + \frac{2x}{v}\right)^2 (1 + \frac{x}{v})^{-4} \left(1 + \frac{2x}{v}\right)^{\frac{v^2}{2}} (1 + \frac{x}{v})^{-v^2}.$$

Now we define $a_v(x)$ and $b_v(x)$ by

$$a_\nu(x) = \left(1 + \tfrac{2x}{\nu}\right)^2 (1 + \tfrac{x}{\nu})^{-4},$$

and

$$b_\nu(x) = \left(1 + \tfrac{2x}{\nu}\right)^{\frac{\nu^2}{2}} (1 + \tfrac{x}{\nu})^{-\nu^2}.$$

We shall first consider the uniform convergence of the sequence of functions $\{a_\nu(x)\}$. Let $\varepsilon > 0$ be given. Since $\lim_{\nu\to\infty} \left(1 \pm \tfrac{2b}{\nu}\right)^2 = 1$ and $\lim_{\nu\to\infty} \dfrac{1}{\left(1 \pm \frac{b}{\nu}\right)^4} = 1$, there exists an integer N such that if $\nu \geq N$, then $-\nu < -b$, $\left|\left(1 \pm \tfrac{2b}{\nu}\right)^2 - 1\right| < \varepsilon$, and $\left|\dfrac{1}{\left(1 \pm \frac{b}{\nu}\right)^4} - 1\right| < \varepsilon$. Then for $0 \leq x \leq b$ and $\nu \geq N$,

$$1 - \varepsilon < \frac{1}{\left(1 + \frac{b}{\nu}\right)^4} \leq \frac{1}{(1 + \frac{x}{\nu})^4} \leq 1,$$

and

$$1 \leq \left(1 + \tfrac{2x}{\nu}\right)^2 < \left(1 + \tfrac{2b}{\nu}\right)^2 < 1 + \varepsilon.$$

Now multiplying the inequalities we have

$$1 - \varepsilon < \frac{\left(1 + \frac{2x}{\nu}\right)^2}{(1 + \frac{x}{\nu})^4} < 1 + \varepsilon,$$

for $0 \leq x \leq b$. Similarly it can be shown that $\dfrac{\left(1 + \frac{2x}{\nu}\right)^2}{(1 + \frac{x}{\nu})^4}$ is within ε of 1 for for $-b \leq x \leq 0$ and $\nu \geq N$. Thus for all $x \in [-b, b]$, $\dfrac{\left(1 + \frac{2x}{\nu}\right)^2}{(1 + \frac{x}{\nu})^4}$ is within ε of 1 if $\nu \geq N$. Therefore the sequence of functions $\{a_\nu(x)\}$ converges uniformly to 1.

We now consider the uniform convergence of the sequence of functions $\{b_\nu(x)\}$. It is our assertion that the sequence of functions $\{b_\nu(x)\}$ is decreasing for

$x > 0$ and increasing for $x < 0$. To demonstrate this result it will suffice to show that the sequence $\{p_v(x)\} = \{\ln(b_v(x))\}$ is decreasing for $x > 0$ and increasing for $x < 0$.

The Maclaurin series expansion of $\ln(1+x)$, for $-1 < x < 1$ is given by $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$ [Section 1.6]. Thus,

$$p_v(x) = \frac{v^2}{2} \ln\left(1 + \frac{2x}{v}\right) - v^2 \ln\left(1 + \frac{x}{v}\right)$$

$$= \frac{v^2}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2x)^k}{k v^k} - v^2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k v^k}$$

$$= \frac{v^2}{2} \left(\frac{2x}{v} - \frac{2x^2}{v^2} + \frac{8x^3}{3v^3} - \frac{4x^4}{v^4} + \frac{32x^5}{5v^5} - \ldots\right)$$
$$- v^2 \left(\frac{x}{v} - \frac{x^2}{2v^2} + \frac{x^3}{3v^3} - \frac{x^4}{4v^4} + \frac{x^5}{5v^5} - \ldots\right)$$

$$= \left(vx - x^2 + \frac{4x^3}{3v} - \frac{2x^4}{v^2} + \frac{16x^5}{5v^3} - \ldots\right)$$
$$- \left(vx - \frac{x^2}{2} + \frac{x^3}{3v} - \frac{x^4}{4v^2} + \frac{x^5}{5v^3} - \ldots\right)$$

$$= -\left(\frac{x^2}{2} - \frac{x^3}{v} + \frac{7x^4}{4v^2} - \frac{3x^5}{v^3} + \ldots\right).$$

Now for $x < 0$, each term in the expression $\left(\frac{x^2}{2} - \frac{x^3}{v} + \frac{7x^4}{4v^2} - \frac{3x^5}{v^3} + \ldots\right)$ is positive since the odd powers of x have negative coefficients. Thus as v increases, the terms become smaller, so the expression decreases with respect to v. Thus the sequence of functions $\{p_v(x)\}$ increases for $x < 0$.

To show that $\{p_v(x)\}$ decreases for $x > 0$, we shall show that its derivative with respect to v is negative. Taking the derivative of $p_v(x)$ and using the Maclaurin series, we obtain:

$$\frac{d(p_v(x))}{dv} = v\ln\left(1+\frac{2x}{v}\right)+\left(\frac{v^2}{2}\right)\left(\frac{1}{1+\frac{2x}{v}}\right)\left(\frac{-2x}{v^2}\right)-2v\ln\left(1+\frac{x}{v}\right)$$
$$-v^2\left(\frac{1}{1+\frac{x}{v}}\right)\left(\frac{-x}{v^2}\right)$$

$$= v\ln\left(1+\frac{2x}{v}\right)-2v\ln\left(1+\frac{x}{v}\right)+\left(\frac{v^2}{2}\right)\left(\frac{-2x}{v(v+2x)}\right)$$
$$-v^2\left(\frac{-x}{v(v+x)}\right)$$

$$= v\left(\frac{2x}{v}-\frac{2x^2}{v^2}+\frac{8x^3}{3v^3}-\frac{4x^4}{v^4}+\frac{32x^5}{5v^5}-\ldots\right)-$$
$$2v\left(\frac{x}{v}-\frac{x^2}{2v^2}+\frac{x^3}{3v^3}-\frac{x^4}{4v^4}+\frac{x^5}{5v^5}-\ldots\right)+\frac{v^2x^2}{v(v+x)(v+2x)}$$

$$= \left(-\frac{x^2}{v}+\frac{2x^3}{v^2}-\frac{7x^4}{2v^3}+\ldots\right)+\frac{v^2x^2}{v(v+x)(v+2x)}.$$

The terms $\frac{(2x)^k-2x^k}{k\,v^{k-1}}$ in the alternating series are decreasing with k, since $0<\frac{x}{v}<1$. Thus the series converges; but more importantly, the entire sum is less than any partial sum that is stopped after a negative term. Therefore for $x>0$,

$$\frac{d(p_v(x))}{dv} < -\frac{x^2}{v}+\frac{v^2x^2}{v(v+x)(v+2x)}$$

$$= \frac{-x^2(v+x)(v+2x)+v^2x^2}{v(v+x)(v+2x)}$$

$$= \frac{-x^2(v^2+3vx+2x^2)+v^2x^2}{v(v+x)(v+2x)}$$

$$= \frac{-2x^4-3vx^3}{v(v+x)(v+2x)}$$

$$< 0.$$

Thus as sequences in v, $\{p_v(x)\}$, and therefore $\{b_v(x)\}$ decreases for $x>0$, increases for $x<0$, and is constant for $x=0$.

By Dini's Theorem [4], the sequence $\{b_v(x)\}$ converges uniformly to $e^{-x^2/2}$ on $[-b, 0]$ and it converges uniformly to $e^{-x^2/2}$ on $[0, b]$. Now for any $\varepsilon > 0$, we choose v large enough so that $\{b_v(x)\}$ is within ε of $e^{-x^2/2}$ for all $x \in [-b, b]$. Therefore the sequence $\{b_v(x)\}$ converges uniformly to $e^{-x^2/2}$ on $[-b, b]$. In Section 1.6 we noted that in general, the product of two uniformly convergent sequences does not necessarily converge uniformly. However if the two uniformly convergent sequences are bounded, we can prove the uniform convergence of the product [1]. Thus we show that $\{U_v(x)\} = \{a_v(x)\, b_v(x)\}$ converges uniformly to $e^{-x^2/2}$.

Since $a_v(x)$ is uniformly convergent, there exists an N such that if $v \geq N$, then $-v < -b$. Thus for $v \geq N$, $a_v(x)$ is continuous, pointwise convergent, and hence bounded on the compact interval $[-b, b]$. Now we have a uniformly convergent sequence of bounded functions. Thus $\{a_v(x)\}$ is uniformly bounded. Hence, there exists a K such that $|a_v(x)| \leq K$ for all $x \in [-b, b]$ and all $v \geq N$.

Now let $\varepsilon > 0$ be given and let $g(x) = e^{-x^2/2}$. Then $|g(x)| \leq 1$. Since $a_v(x)$ and $b_v(x)$ are uniformly convergent, there exists an $N_0 \geq N$ such that if $v \geq N_0$, then $\left|b_v(x) - g(x)\right| < \frac{\varepsilon}{2K}$ and $\left|a_v(x) - 1\right| < \frac{\varepsilon}{2}$ for all $x \in [-b, b]$. Thus for $v \geq N_0$, and all $x \in [-b, b]$, we have the following:

$$
\begin{aligned}
|a_v(x)\, b_v(x) - g(x)| &= |a_v(x)\, b_v(x) - a_v(x)\, g(x) + a_v(x)\, g(x) - g(x)| \\
&\leq |a_v(x)\, b_v(x) - a_v(x)\, g(x)| + |a_v(x)\, g(x) - g(x)| \\
&= |a_v(x)|\ |b_v(x) - g(x)| + |a_v(x) - 1|\ |g(x)|
\end{aligned}
$$

$$\leq (K)\,|\,b_v(x) - g(x)\,| + |\,a_v(x) - 1\,|\,(1)$$

$$< (K)\left(\tfrac{\varepsilon}{2K}\right) + \left(\tfrac{\varepsilon}{2}\right)(1)$$

$$= \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2}$$

$$= \varepsilon.$$

Therefore the sequence $\{U_v(x)\} = \{a_v(x)\, b_v(x)\}$ and also our original sequence $\{U_n(x)\}$ converges uniformly to $e^{-x^2/2}$ on $[-b, b]$. ■

We now consider the sequence of functions $\{T_n(x)\}$.

Lemma 2.5. On any interval $[-b, b]$, the sequence of functions $\{T_n(x)\} = \left\{\left(1 + \frac{2x}{\sqrt{n+4}}\right)^{\frac{n}{2}} \left(1 + \frac{x}{\sqrt{n+4}}\right)^{-(n+4)}\right\}$ converges uniformly to $e^{-x^2/2}$.

Proof. First we show that $T_n(x)$ converges pointwise to $e^{-x^2/2}$. For any interval $[-b, \infty)$,

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \left(1 + \frac{2x}{\sqrt{n+4}}\right)^{\frac{n}{2}} \left(1 + \frac{x}{\sqrt{n+4}}\right)^{-(n+4)}$$

$$= \lim_{n\to\infty} \left(1 + \frac{2}{\sqrt{n}}\; x\right)^{\frac{n}{2}} \left(1 + \frac{1}{\sqrt{n}}\; x\right)^{-n}$$

$$= e^{-\frac{x^2}{2}}.$$

[See argument in Theorem 2.3]. Now by letting $w = \sqrt{n+4}$, it will suffice to show that the sequence of functions $\{T_w(x)\} = \left\{\left(1 + \frac{2x}{w}\right)^{\frac{w^2-4}{2}} (1 + \frac{x}{w})^{-w^2}\right\}$ converges uniformly to $e^{-x^2/2}$, as w tends to infinity. We initially start our sequences with n large enough so that $-\sqrt{n+4} < -b$. Thus for our new sequence written in terms of w, we have that $w > b$. Thus for all $[-b, b]$, we have that $-1 < \frac{x}{w} < 1$. It is also the case that this new sequence still converges pointwise to $e^{-x^2/2}$.

To show that the sequence of functions $\{T_w(x)\}$ converges uniformly to $e^{-x^2/2}$, we shall divide $\{T_w(x)\}$ into two pieces. Thus we rewrite $\{T_w(x)\}$ as follows:

$$T_w(x) = \left(1 + \frac{2x}{w}\right)^{-2} \left(1 + \frac{2x}{w}\right)^{\frac{w^2}{2}} (1 + \frac{x}{w})^{-w^2}.$$

Now we define $r_w(x)$ and $s_w(x)$ by

$$r_w(x) = \left(1 + \frac{2x}{w}\right)^{-2},$$

and

$$s_w(x) = \left(1 + \frac{2x}{w}\right)^{\frac{w^2}{2}} (1 + \frac{x}{w})^{-w^2}.$$

From the argument in Lemma 2.4, we know that the sequence of funcitons $\{s_w(x)\} = \left(1 + \frac{2x}{w}\right)^{\frac{w^2}{2}} (1 + \frac{x}{w})^{-w^2}$ converges uniformly to $e^{-x^2/2}$. Now we consider the uniform convergence of the sequence of functions $\{r_w(x)\}$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{w\to\infty} \frac{1}{\left(1+\frac{2b}{w}\right)^2} = 1$ and $\lim\limits_{w\to\infty} \frac{1}{\left(1-\frac{2b}{w}\right)^2} = 1$, there exists an integer N such that if $w \geq N$, then $-w < -b$, $\left| \frac{1}{\left(1+\frac{2b}{w}\right)^2} - 1 \right| < \varepsilon$, and $\left| \frac{1}{\left(1-\frac{2b}{w}\right)^2} - 1 \right| < \varepsilon$. Then for all $w \geq N$,

$$1 - \varepsilon < \frac{1}{\left(1+\frac{2b}{w}\right)^2} \leq \frac{1}{\left(1+\frac{2x}{w}\right)^2} \leq 1,$$

for $0 \le x \le b$; and

$$1 \le \frac{1}{\left(1+\frac{2x}{w}\right)^2} \le \frac{1}{\left(1-\frac{2b}{w}\right)^2} < 1+\varepsilon,$$

for $-b \le x \le 0$. Thus for all $x \in [-b, b]$, $\dfrac{1}{\left(1+\frac{2x}{w}\right)^2}$ is within ε of 1 if $w \ge N$.

Therefore the sequence of functions $\{r_w(x)\}$ converges uniformly to 1.

We have shown that the separate pieces of $\{T_w(x)\}$ are uniformly convergent. We now show that the product converges uniformly to $e^{-x^2/2}$. Since $r_w(x)$ is uniformly convergent, there exists an N such that if $w \ge N$, then $-w < -b$. Thus for $w \ge N$, $r_w(x)$ is continous, pointwise convergent, and hence bounded on the compact interval $[-b, b]$. Now we have a uniformly convergent sequence of bounded functions. Thus $\{r_w(x)\}$ is uniformly bounded. Hence, there exists a B such that $|r_w(x)| \le B$ for all $x \in [-b, b]$ and all $w \ge N$.

Now let $\varepsilon > 0$ be given and let $g(x) = e^{-x^2/2}$. Then $|g(x)| \le 1$. Since $r_w(x)$ and $s_w(x)$ are uniformly convergent, there exists an $N_1 \ge N$ such that if $w \ge N_1$, then $\left|s_w(x) - g(x)\right| < \frac{\varepsilon}{2B}$ and $\left|r_w(x) - 1\right| < \frac{\varepsilon}{2}$ for all $x \in [-b, b]$.

Thus for $w \ge N_1$, and all $x \in [-b, b]$, we have the following:

$$\begin{aligned}
|r_w(x)\, s_w(x) - g(x)| &= |r_w(x)\, s_w(x) - r_w(x)\, g(x) + r_w(x)\, g(x) - g(x)| \\
&\le |r_w(x)\, s_w(x) - r_w(x)\, g(x)| + |r_w(x)\, g(x) - g(x)| \\
&= |r_w(x)|\ |s_w(x) - g(x)| + |r_w(x) - 1|\ |g(x)| \\
&\le (B)\,|s_w(x) - g(x)| + |r_w(x) - 1|\ (1) \\
&< (B)\left(\tfrac{\varepsilon}{2B}\right) + \left(\tfrac{\varepsilon}{2}\right)(1)
\end{aligned}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

Therefore the sequence $\{T_w(x)\} = \{r_w(x)\, s_w(x)\}$ and also our original sequence $\{T_n(x)\}$ converges uniformly to $e^{-x^2/2}$ on $[-b, b]$. ■

We are now ready to show the uniform convergence of $\{k_{m,n}(x)\}$ when $m = n$ in the following lemma.

Lemma 2.6. On any interval $[-b, b]$, the sequence of functions

$$\{k_{m,n}(x)\} = \left\{ \left(1 + \sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\; x\right)^{\frac{m}{2}} \left(1 + \sqrt{\frac{2\,m}{(m+n-2)\,(n-4)}}\; x\right)^{-\frac{m+n}{2}} \right\}$$

converges uniformly to $g(x) = e^{-x^2/2}$, when $m = n$.

Proof. Let $g(x) = e^{-x^2/2}$ and let $m = n$. Then $k_{m,n}(x)$ becomes:

$$k_n(x) = \left(1 + 2\,x \sqrt{\frac{n-1}{n\,(n-4)}}\right)^{\frac{n}{2}} \left(1 + x \sqrt{\frac{n}{(n-1)\,(n-4)}}\right)^{-n}.$$

Now let $\varepsilon > 0$ be given. By Lemmas 2.4 and 2.5 there exists an N such that if $n \geq N$, then $|\, T_n(x) - g(x)\,| < \varepsilon$ and $|\, U_n(x) - g(x)\,| < \varepsilon$ for all $x \in [-b, b]$. This implies that $g(x) - \varepsilon < T_n(x) < g(x) + \varepsilon$ and $g(x) - \varepsilon < U_n(x) < g(x) + \varepsilon$, for all $x \in [-b, b]$ and $n \geq N$. Now by Proposition 2.1 we have $T_n(x) \leq k_n(x) \leq U_n(x)$ for $x \geq 0$, and $U_n(x) \leq k_n(x) \leq T_n(x)$, for $x \leq 0$. Thus for $n \geq N$ and $x \in [0, b]$,

$$g(x) - \varepsilon < T_n(x) \leq k_n(x) \leq U_n(x) < g(x) + \varepsilon\,,$$

which implies that

$$g(x) - \varepsilon < k_n(x) < g(x) + \varepsilon .$$

Thus $|\, k_n(x) - g(x)\,| < \varepsilon$ for $n \geq N$ and for all $x \in [0, b]$. Now for $n \geq N$ and $x \in [-b, 0]$, we have

$$g(x) - \varepsilon < U_n(x) \leq k_n(x) < T_n(x) \leq g(x) + \varepsilon ,$$

which implies that

$$g(x) - \varepsilon < k_n(x) < g(x) + \varepsilon .$$

Thus $|\, k_n(x) - g(x)\,| < \varepsilon$ for $n \geq N$ and for all $x \in [-b, 0]$. Therefore the sequence of functions $\{k_n(x)\}$ converges uniformly to $e^{-x^2/2}$ on $[-b, b]$. ■

As previously stated, the product of two uniformly convergent sequences converges uniformly if the sequences are bounded. Thus we have the following lemma.

Lemma 2.7. On any interval $[-b, b]$, the sequence of functions

$$\left\{\frac{1}{1+\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;x}\left(1+\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;x\right)^{\frac{m}{2}}\left(1+\sqrt{\frac{2\,m}{(m+n-2)\,(n-4)}}\;x\right)^{-\frac{m+n}{2}}\right\}$$

converges uniformly to $e^{-x^2/2}$, when $m = n$.

Proof. Let $g(x) = e^{-x^2/2}$, $h_{m,n}(x) = \dfrac{1}{1+\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;x}$, and

$$k_{m,n}(x) = \left(1+\sqrt{\frac{2\,(m+n-2)}{m\,(n-4)}}\;x\right)^{\frac{m}{2}}\left(1+\sqrt{\frac{2\,m}{(m+n-2)\,(n-4)}}\;x\right)^{-\frac{m+n}{2}} .$$ Now let

$m = n$. Then $h_{m,n}(x)$ becomes:

$$h_{m,n}(x) = \frac{1}{1+2\,x\sqrt{\frac{n-1}{n\,(n-4)}}\;x},$$

and $k_{m,n}(x)$ becomes:

$$k_n(x) = \left(1+2\,x\sqrt{\frac{n-1}{n\,(n-4)}}\right)^{\frac{n}{2}}\left(1+x\sqrt{\frac{n}{(n-1)\,(n-4)}}\right)^{-n}.$$

Clearly $|g(x)| \le 1$ for all x. As in Lemma 2.3, there exists an N such that if $n \ge N$, then $-\frac{1}{2}\sqrt{\frac{n(n-4)}{n-1}} < -b$. Thus for $n \ge N$, $h_n(x)$ is continuous, pointwise convergent, and hence bounded on the compact interval $[-b, b]$. Now we have a uniformly convergent sequence of bounded functions. Thus $\{h_n(x)\}$ is uniformly bounded. Hence, there exists an W such that $|h_n(x)| \le W$ for all $x \in [-b, b]$ and all $n \ge N$ [1].

Now let $\varepsilon > 0$ be given. By Lemmas 2.3 and 2.6 there exists an $N_0 \ge N$ such that if $n \ge N_0$, then $\left|k_n(x) - g(x)\right| < \frac{\varepsilon}{2W}$ and $\left|h_n(x) - 1\right| < \frac{\varepsilon}{2}$ for all $x \in [-b, b]$. Thus for $n \ge N_0$, and all $x \in [-b, b]$, we have the following:

$$\begin{aligned}
|h_n(x)\,k_n(x) - g(x)| &= |h_n(x)\,k_n(x) - h_n(x)\,g(x) + h_n(x)\,g(x) - g(x)| \\
&\le |h_n(x)\,k_n(x) - h_n(x)\,g(x)| + |h_n(x)\,g(x) - g(x)| \\
&= |h_n(x)|\;|k_n(x) - g(x)| + |h_n(x) - 1|\;|g(x)| \\
&\le (W)\,|k_n(x) - g(x)| + |h_n(x) - 1|\;(1) \\
&< (W)\left(\tfrac{\varepsilon}{2W}\right) + \left(\tfrac{\varepsilon}{2}\right)(1) \\
&= \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therfore the sequence of functions $\{h_{m,n}(x)\, k_{m,n}(x)\}$ converges uniformly to $e^{-x^2/2}$ on $[-b, b]$, when $m = n$. ■

We are now ready to prove our main result.

Theorem 2.4. On any interval $[-b, b]$, the probability density functions $\{g_{m,n}(x)\}$ of the standardized F-distributions converge uniformly to the probability density function $f(x)$ of the standard normal distribution, when $m = n$.

Proof. Let $m = n$. Then we write $g_n(x) = C_n\, h_n(x)\, k_n(x)$ and $f(x) = C\, g(x)$, where the product of the sequence of functions $\{h_n(x)\, k_n(x)\}$ converges uniformly to $g(x)$, and the sequence of constants $\{C_n\}$ converges to C. The sequence $\{C_n\}$ is bounded by some constant R, since $\{C_n\}$ converges to C [4]. As noted in Lemma 2.7, $g(x)$ is bounded by 1. Now given $\varepsilon > 0$, there exists an N such that if $n \geq N$, then $\left| C_n - C \right| < \frac{\varepsilon}{2}$ and $\left| h_n(x)\, k_n(x) - g(x) \right| < \frac{\varepsilon}{2R}$ for all $x \in [-b, b]$. Then for $n \geq N$, and all $x \in [-b, b]$, we have the following:

$$\begin{aligned}
| g_n(x) - f(x) | &= | C_n\, h_n(x)\, k_n(x) - C\, g(x) | \\
&= | C_n\, h_n(x)\, k_n(x) - C_n\, g(x) + C_n\, g(x) - C\, g(x) | \\
&\leq | C_n\, h_n(x)\, k_n(x) - C_n\, g(x) | + | C_n\, g(x) - C\, g(x) | \\
&= | C_n | \; | h_n(x)\, k_n(x) - g(x) | + | C_n - C | \; | g(x) | \\
&< (R)\left(\tfrac{\varepsilon}{2R}\right) + \left(\tfrac{\varepsilon}{2}\right)(1) \\
&= \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore the p.d.f.'s $\{g_{m,n}(x)\}$ of the standardized F-distributions converge uniformly to the p.d.f $f(x)$ of the standard normal distribution when $m = n$. ∎

We can pose the question do the probability density functions $\{g_{m,n}(x)\}$ of the standardized F-distributions converge uniformly to the probablility density function $f(x)$ of the standard normal distribution when $m \neq n$? It is possible to use the previous argument to show that the sequence of functions $\{g_{m,n}(x)\}$ converges uniformly to $f(x)$ when $m > n$, that is for $m = n + p$, where $p \geq 0$ if we can show that $T_n(x) \leq k_{n+p}(x) \leq U_n(x)$ for $x \geq 0$, and $U_n(x) \leq k_{n+p}(x) \leq T_n(x)$, for $x \leq 0$. Figures 15 and 16 indicate that it is possible to show analytically that this relationship holds for n, $p > 0$.

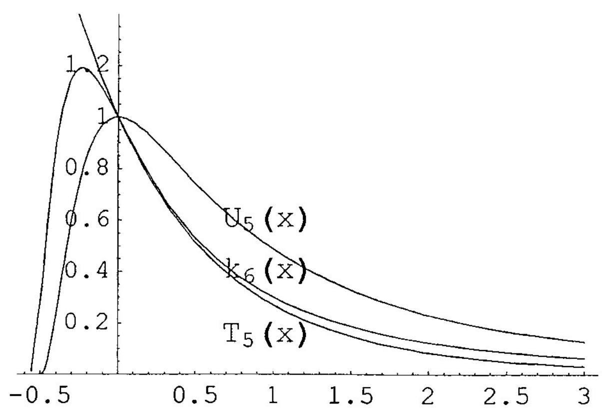

Figure 15: $T_n(x)$, $k_{n+p}(x)$, and $U_n(x)$ when $n = 5$ and $p = 1$.

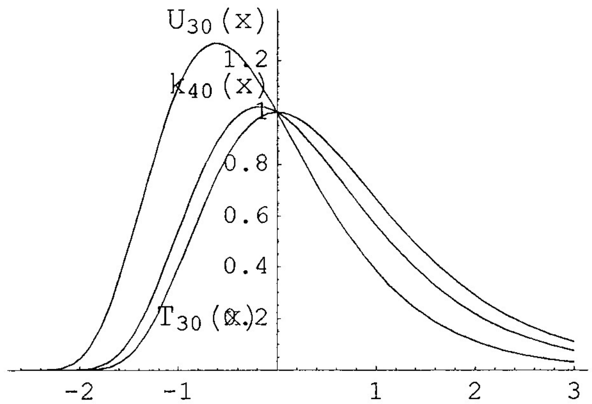

Figure 16: $T_n(x)$, $k_{n+p}(x)$, and $U_n(x)$ when $n = 30$ and $p = 10$.

Figures 17 and 18 examine the relationship between $\{g_{m,n}(x)\}$ and $f(x)$ when $m < n$.

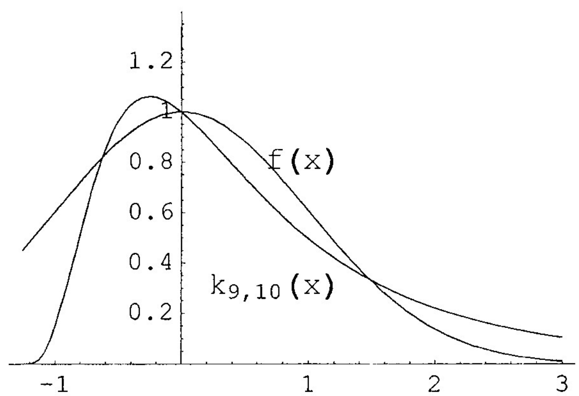

Figure 17: $f(x)$ and $k_{m,n}(x)$ when $m = 9$ and $n = 10$.

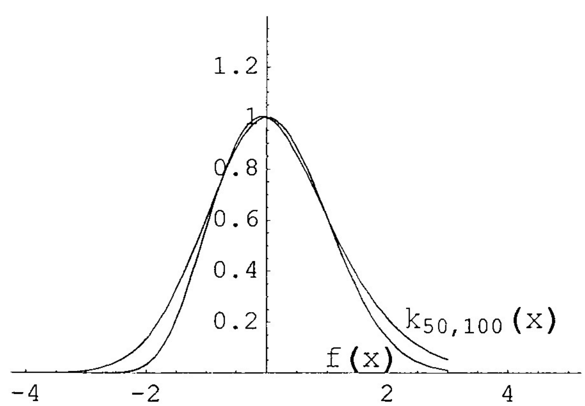

Figure 18: $f(x)$ and $k_{m,n}(x)$ when $m = 100$ and $n = 50$.

These graphs suggest that $\{g_{m,n}(x)\}$ converges uniformly to $f(x)$ when $m < n$. Thus it is conceivable that one can show that the probability density functions of the standardized F-distributions converge uniformly to the probablility density function of the standard normal distribution when $m < n$.

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